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Numerical Analysis

An Aitken-like acceleration method applied to missing boundary data reconstruction for the Cauchy–Helmholtz problem

Une méthode d'accélération de type Aitken appliquée à la reconstruction de données frontières manquantes sur le problème de Cauchy–Helmholtz

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A R T I C L E   I N F O

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A B S T R A C T

This Note is concerned with the severely ill-posed Cauchy–Helmholtz problem. This Cauchy problem being rephrased through an "interfacial" equation, we resort to an Aitken–Schwarz method for solving this equation. Numerical trials highlight the efficiency of the present method.

R É S U M É

Cette Note concerne le problème mal-posé de Cauchy–Helmholtz. Ce problème est interprété en terme d'équation d'interface qu'on résout via une méthode d'Aitken–Schwarz. Des essais numériques illustrent l'efficacité de cette méthode.

Version française abrégée

Cette Note présente une méthode pour résoudre le problème de complétion des données pour l'équation de Helmholtz (i.e. problème de Cauchy). Ce problème est connu pour être mal posé au sens de Hadamard. On donne un flux φ et une pression f sur une partie de la frontière surdéterminée Γ_c d'un domaine Ω et on veut compléter les données sur l'autre partie de la frontière inconnue Γ_i. On montre que le problème de Cauchy peut être reécrit en terme d'opérateur de Steklov–Poincaré écrit sur l'interface de la partie à compléter.

Koslov et al. (KMF) [9] ont proposé une méthode itérative pour résoudre les problèmes de Cauchy, qui s'avère être un algorithme de Richardson préconditionné pour résoudre une équation sur l'interface. Cet algorithme appliqué à l'équation de Helmholtz diverge purement linéairement. Ceci nous a permis d'accélérer la convergence de l'algorithme par la technique d'Aitken–Schwarz proposée dans [7].

La validation de cette méthode est testée sur l'exemple analytique $u = e^{2ixy}$. Dans le but de tester la robustesse de notre algorithme nous avons bruité les données. La Fig. 1 illustre la répartition de la pression calculée sur la frontière à compléter pour des données bruitées et non bruitées. Le Tableau 1 illustre l'erreur de complétion pour différents niveaux de bruit blanc.

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1. Introduction

This contribution is concerned with the recovering of both Dirichlet and Neumann data on some part of the domain boundary, starting from the knowledge of these data on another part of the boundary for the Helmholtz equation. This data completion question may be relevant by itself in some practical applications or maybe a preliminary step to others. For example, the reconstruction of acoustic pressure fields inside the cavity of a vibrating object by Wu and Yu [15]. The Cauchy problem under consideration is known to be ill-posed in so far as the existence of a solution to such a problem requires that compatibility conditions should be insured. Furthermore, the data completion problem is highly instable. For the theoretical issues we refer to [12] and references therein. In a classical way, one can resort to a least square type method (misfit computation-measurement) in solving such an inverse problem. Notice that, regarding the ill-posedness of the problem under consideration, this method should be associated to some regularizing one [3]. However, our approach, in this Note, is different, it can be linked to [9].

Koslov et al. (KMF) [9] proposed an alternating iterative method for solving Cauchy problems which turns out to be a Richardson preconditioned algorithm for solving an interface equation, see [1], and references therein applied to the Helmholtz equation. The KMF's algorithm, in its initial formulation does not converge, indeed, it diverges linearly. In a recent theoretical paper, Johansson and Kozlov [8] proposed a KMF [9] modified alternating procedure for solving Cauchy problems for self-adjoint non-coercive elliptic operators. Our approach, in this Note, is quite different: since the KMF's algorithm applied to Helmholtz equation diverges linearly, we apply the Aitken-like acceleration process.

The continuous problem of the data completion for the Helmholtz operator is formulated as follows:

Let Ω be a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. The boundary $\Gamma = \partial\Omega$, assumed smooth, is split into Γ_c and Γ_i having both non-vanishing measure, whose outer normal direction is denoted by $\mathbf{n}$. Given a flux φ and the data f on the overdetermined boundary Γ_c, recovering the data on the remainder (incomplete) part Γ_i of the boundary is accomplished by solving the Cauchy system that may be put under the following mathematical setting: *find u such that*

$$\begin{cases} \Delta u + k^2 u = g & \text{in } \Omega, \\ \partial_n u = \varphi & \text{on } \Gamma_c, \\ u = f & \text{on } \Gamma_c. \end{cases} \tag{1}$$

This Note is outlined as follows: In the opening section, the Cauchy–Helmholtz problem is rephrased in terms of an interfacial problem using the Steklov–Poincaré operator, the Aitken–Schwarz acceleration process is described. Section 3 is devoted to the numerical illustration. The closing section is devoted to some comments.

2. Data completion process

We have a double condition on Γ_c, let λ be an auxiliary field defined on Γ_i, we introduce two well-defined boundary value problems having a Dirichlet data on Γ_i equal to λ. For the remaining part of the boundary Γ_c, we can combine different choices for boundary conditions (Dirichlet, Neumann or Robin). We impose the Robin type boundary condition on Γ_c because it is necessary to insure the well-posedness of the near field Helmholtz problem. We consider therefore the two following Helmholtz problems:

Find $v(\lambda, \varphi + iqf)$ and $w(\lambda, \varphi + iq'f)$ solutions of

$$\begin{cases} \Delta v(\lambda) + k^2 v(\lambda) = g & \text{in } \Omega, \\ \partial_n v(\lambda) + iqv(\lambda) = \varphi + iqf & \text{on } \Gamma_c, \\ v(\lambda) = \lambda & \text{on } \Gamma_i, \end{cases} \qquad \begin{cases} \Delta w(\lambda) + k^2 w(\lambda) = g & \text{in } \Omega, \\ \partial_n w(\lambda) + iq'w(\lambda) = \varphi + iq'f & \text{on } \Gamma_c, \\ w(\lambda) = \lambda & \text{on } \Gamma_i. \end{cases} \tag{2}$$

Here q and q' are a real constant ($|q| \neq |q'|$).

Remark. We selected these mixed-value problems with Robin condition on Γ_c to insure the existence and uniqueness of the forward problem and to avoid resonant frequencies.

Solving the Cauchy system (1) is achieved when the data extension λ makes v and w coincide, and the solution is then $u = v = w$. This leads to write an equation on Γ_i to be satisfied by λ:

$$\frac{\partial v(\lambda, \varphi + iqf)}{\partial n} = \frac{\partial w(\lambda, \varphi + iq'f)}{\partial n}. \tag{3}$$

One poses $v(\lambda, \varphi + iqf) = v(\lambda, 0) + v(0, \varphi + iqf) = v^0(\lambda) + v^*$ and $w(\lambda, \varphi + iq'f) = w(\lambda, 0) + w(, \varphi + iq'f) = w^0(\lambda) + w^*$.$v^0$ and w^0 are the Helmholtz-free extensions of λ from Γ_i into Ω, noted respectively $H_v(\lambda)$ and $H_w(\lambda)$. Whereas v^* and w^* are two Helmholtz-free extensions of $(\varphi + iqf)$ and $(\varphi + iq'f)$ from Γ_i into Ω, noted respectively $R_v(\varphi + iqf)$ and $R_w(\varphi + iq'f)$.

The latter condition amounts to the requirement that λ satisfies the Steklov–Poincaré type equation

$$S\lambda = \chi \quad \text{on } \Gamma_i \tag{4}$$

where

$$\chi := -\partial_n R_v(\varphi + iqf) + \partial_n R_w(\varphi + iq'f)$$

and S is the Helmholtz–Cauchy–Steklov–Poincaré operator formally defined by

$$S\lambda := S_v - S_w = \partial_n H_v(\lambda) - \partial_n H_w(\lambda).$$

As pointed out in [1], the algorithms of Richardson

$$\lambda^{n+1} = \lambda^n + (S\lambda^n - \chi) \tag{5}$$

and of KMF (Richardson preconditioned with S_-) failed to converge. Notice that the KMF's algorithm is very long to converge in the elliptic framework and it is no more convergent for the Helmholtz case.

2.1. *The Aitken–Cauchy algorithm*

The aim of this Note is to "force" the convergence of the KMF algorithm in the Helmholtz case resorting to an Aitken-like acceleration procedure introduced in [6,7] for the Domain Decomposition method with Dirichlet–Dirichlet "or Dirichlet–Neumann" boundary condition.

The KMF's algorithm search for λ^∞ reads as solving the interface equation:

$$\lambda^{n+1} = \lambda^n + S_-^{-1}(S\lambda^n - \chi) = S_-^{-1}(S_+\lambda^n - \chi). \tag{6}$$

The main observation to build the Aitken–Schwarz methodology is to remark that the convergence of the error and the normal derivative of the error at the artificial interface behaves linearly:

$$(\lambda^{n+1} - \lambda^\infty) = (I + S_-^{-1}S)(\lambda^n - \lambda^\infty) = S_-^{-1}S_+(\lambda^n - \lambda^\infty), \tag{7}$$

$$S_-(\lambda^{n+1} - \lambda^\infty) = S_+S_-^{-1}(S_-(\lambda^{n+1} - \lambda^\infty)). \tag{8}$$

The Aitken–Schwarz methodology [6,7,5,14,4] consists on building a cheap approximation of the operator

$$P = \begin{pmatrix} S_-^{-1}S_+ & 0 \\ 0 & S_+S_-^{-1} \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

based only on the interface solution and derivative solution iterates unlike to the approach of [11] where an a priori knowledge of the operator is needed.

Let be $\mu = (\lambda, S_-\lambda)^t$. Assume that the sequence $(\mu^n)_{n\in\mathbf{N}}$ converges linearly toward $\mu^\infty \in K^N$, $K = \{\mathbb{R}, \mathbb{C}\}$ in the meaning of Eqs. (7), (8) with a constant full rank error operator P independent of n. Assume that there is a norm $\|.\|$ such that $\|P\| < 1$. Then P and μ^∞ can be determined from several iterates using the equations

$$(\mu^{N+1} - \mu^N, \ldots, \mu^2 - \mu^1) = P(\mu^N - \mu^{N-1}, \ldots, \mu^1 - \mu^0). \tag{9}$$

If $Id - P$ is non-singular ($\|P\| < 1$ for example), then μ^∞ can be deduced as

$$\mu^\infty = (Id - P)^{-1}(\mu^{m+1} - P\mu^m), \quad \forall m \geqslant 1. \tag{10}$$

The construction of P requires at least $N+1$ iterates if the error components are linked together. The nature of the convergence does not change if we consider the error coefficients $e^n = \mu^n - \mu^\infty$ in an orthogonal basis Φ (named "Fourier basis" with the property to have a decreasing of coefficients in this basis) instead of the original basis (named physical space). Then we can write an equivalent equation rather than Eqs. (7), (8). Let us consider $\hat\beta^n_{\Gamma_i}$ the components of the traces of $E^n = \mu^{n+1} - \mu^n$ in the "Fourier" basis Φ. Then one can write:

$$\hat\beta^n_{\Gamma_i} = P_{[[.,.]]}\hat\beta^{n+1}_{\Gamma_i}. \tag{11}$$

This matrix $P_{[[.,.]]}$ has the same size as the matrix P. Nevertheless, we have more flexibility to define some consistent approximation of this matrix, since we have access to *a posteriori* estimate based on the module value of the Fourier coefficients. We derive then the adaptive Aitken–Schwarz algorithm as follows [4]:

Algorithm 2.1.

(i) *Perform q iterations of the KFM (Schwarz) algorithm.*
(ii) *Write the difference between two successive iterates in the Φ basis and select the component modes higher than a fixed tolerance or fixed relative tolerance with respect to the highest mode if the Schwarz diverges.*
(iii) *Take the subset $\tilde\mu$ of m Fourier modes from* 1 *to* $\min(q, \max(Index))$.
(iv) *Compute the $m \times m$ $P^*_{[[.,.]]}$ matrix which columns are associated to the m modes selected which is an approximation of $P_{[[.,.]]}$.*
(v) *Accelerate the m modes with the Aitken formula*: $\tilde\mu^\infty = (Id - P^*_{[[.,.]]})^{-1}(\tilde\mu^{n+1} - P^*_{[[.,.]]}\tilde\mu^n)$.
(vi) *Recompose the solution with the m modes accelerated and with the $N - m$ other modes if the Schwarz converges else put them to* 0.

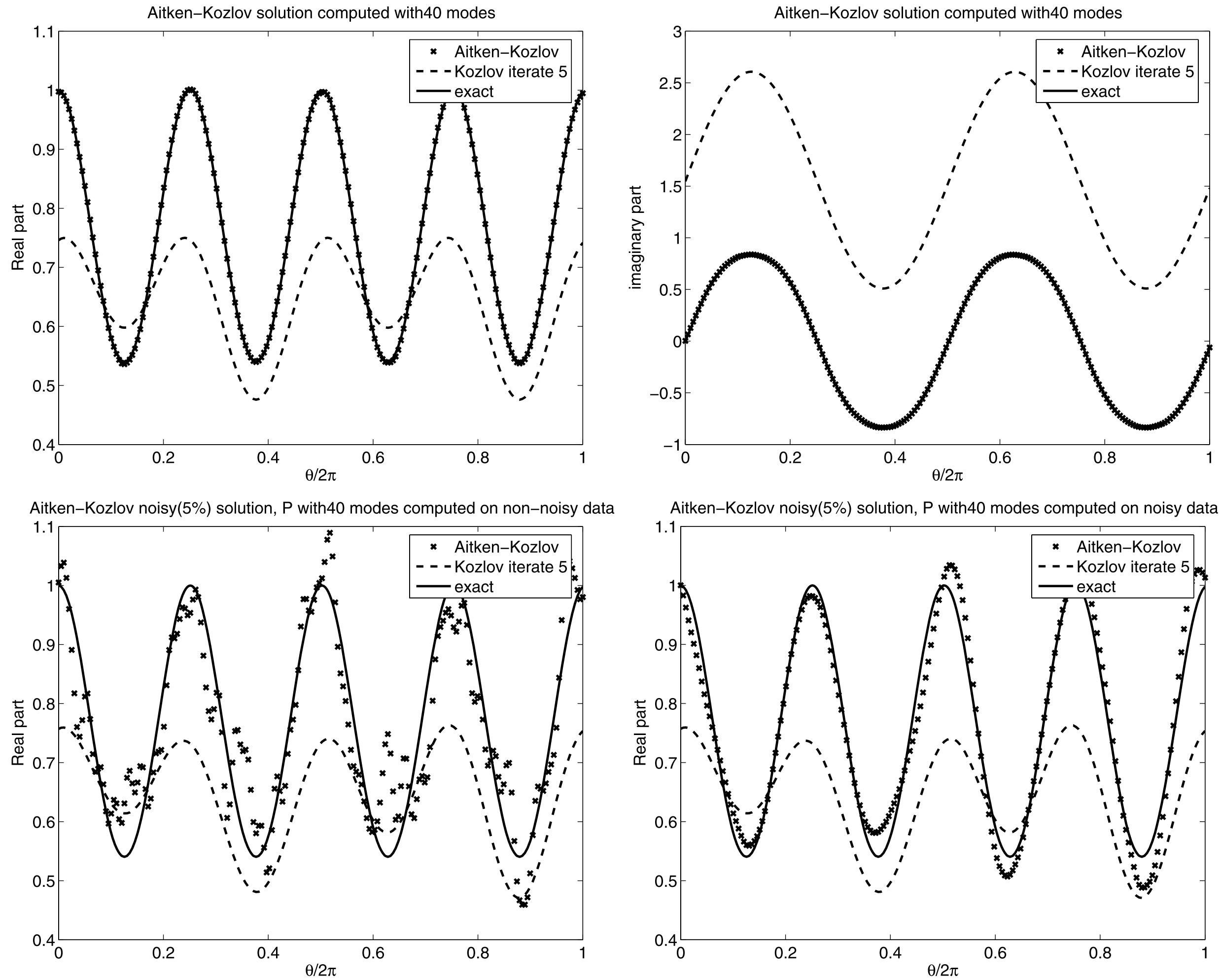

Fig. 1. Reconstruction on the internal boundary of a tube, comparison between exact and Aitken–Kozlov solution of the acoustic pressure computed with no noise: real part (left top), imaginary part (right top) and the real part with 5% noise where $P^*_{[[.,.]]}$ is build with no noisy data (left bottom) and with noisy data (right bottom).

3. Numerical illustration

To conduct the numerical trials we use the Finite Element **MELINA** code [10] for solving the forward problems (techniques developed in [2] could also be used for performance considerations) and Matlab for the acceleration process. Ω is considered as a thick annular domain with radii $r_1 = 1$ defining Γ_i and $r_2 = 1.5$ defining Γ_c. The searched data is the harmonic function $u = e^{i2xy}$. The overdetermined boundary conditions on Γ_c are $f = u$ and $\varphi = \partial_n u$. The boundary conditions on Γ_i have to be rebuilt by the Aitken–KMF's process. The meshes we use are triangular, the finite elements are linear. The calculations are run on a uniform mesh with 200 nodes on Γ_i, 300 nodes on Γ_c and 5173 nodes on Ω. It is noted that the relaxed KMF's algorithm ($\lambda^{n+1} = (1-r)\lambda^n + rS_-^{-1}(S\lambda^n - \chi)$ with $r = 0.4$) exhibits a linear divergence. The Fourier coefficient q (respectively q') defined in (2) is fixed to 1 (respectively 5).

Fig. 1 gives the accuracy result between the exact solution and the solution obtained by the Aitken acceleration, based on the divergent solutions of the relaxed KMF's algorithm. We proceed as described in Algorithm 2.1 for the first 40 Fourier positive and negative modes (of the solution and normal derivative) with computing a $P_{[[.,.]]}$ matrix separately for each set. To emphasize the reliability of this algorithm and to attest the stabilizing effect, we performed a reconstruction of the solution from some noisy data. We computed also $P_{[[.,.]]}$ matrices based on noisy and non-noisy data.

Table 1 illustrates the convergence speed and the accuracy of this algorithm for polluted Dirichlet data with different white noise levels 2.5%, 5%, 10%. It shows that the relative error behaves linearly with respect to the noise level and still reasonable with an error near the value of the noise.

From the theoretical view point, the interface operator P does not depend on the loading. It depends only on the geometry. However, at the discrete level, one has do take into account the errors of discretization, approximation as well as to add the errors on the resolution of the local problems.

Table 1
Accuracy on the computational domain of the Aitken–Kozlov solution associated to polluted Dirichlet data with different noise levels of an exact solution.

Noise level	P built with no noisy data		P built with noisy data	
	$\frac{\|u_{ex}-u_{cal}\|_{L^2(\Omega)}}{\|u_{ex}\|_{L^2(\Omega)}}$	$\frac{\|u_{ex}-u_{cal}\|_{L^\infty(\Omega)}}{\|u_{ex}\|_{L^\infty(\Omega)}}$	$\frac{\|u_{ex}-u_{cal}\|_{L^2(\Omega)}}{\|u_{ex}\|_{L^2(\Omega)}}$	$\frac{\|u_{ex}-u_{cal}\|_{L^\infty(\Omega)}}{\|u_{ex}\|_{L^\infty(\Omega)}}$
2.5%	2.8×10^{-2}	1.29×10^{-1}	1.5×10^{-2}	3.2×10^{-2}
5%	5.6×10^{-2}	2.42×10^{-1}	2.27×10^{-2}	7.64×10^{-2}
10%	11.25×10^{-2}	4.5×10^{-1}	4.67×10^{-2}	1.59×10^{-1}

4. Conclusions

This Note deals with a method to solve the Cauchy problem for the Helmholtz equation. The KMF's algorithm is known to diverge in this problem. Nevertheless, the pure linear divergence allows to apply the Aitken acceleration of the convergence process to get the converged solution. The building of the matrix of acceleration takes advantages of the decomposition of the trace solution on the boundary where data are missing, in an orthogonal basis for which the values of components decrease with the number of mode, in order to have a cheap approximation of linear operator of the error at artificial interface. This operator is linked to the Dirichlet–Neumann mapping, but no direct information on the operator is used to build an approximation of it.

Up to our knowledge there are very few effective results for reconstruction problem for Helmholtz equation except for the case where the data are lacking on a flat boundary [13]. Presently further numerical experiments are going on as well as some practical applications such as the interfacial crack identification problem.

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