## Complex Analysis

# Surjectivity criteria for convolution operators in $A^{-\infty}$ 

# Critères de surjectivité pour des opérateurs de convolution dans $A^{-\infty}$ 

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#### Abstract

The goal of this Note is to prove criteria for surjectivity of convolution operators acting from $A^{-\infty}(\Omega+K)$ into $A^{-\infty}(\Omega)$ ( $\Omega$ and $K$ being a bounded convex domain and a convex compact set in $\mathbb{C}^{n}(n>1)$, respectively $)$. This is obtained in a connection with the division problem. The explicit representation of solutions of the corresponding convolution equations in a form of Dirichlet series is also given.


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## R É S U M É

Le but de cet article est d'établir des critères de surjectivité pour des opérateurs de convolution, opérant de $A^{-\infty}(\Omega+K)$ dans $A^{-\infty}(\Omega)$ ( $\Omega$ et $K$ étant, respectivement, un domaine convexe borné et un compact convexe dans $\left.\mathbb{C}^{n}(n>1)\right)$. Ils seront obtenus en les reliant au problème de division. Une représentation explicite des solutions des équations de convolution correspondantes sera également donnée sous forme de série de Dirichlet.
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## 1. The main goal and result

### 1.1. Basic notations

$\mathcal{O}(\Omega)$ denotes the space of functions holomorphic in a domain $\Omega \subset \mathbb{C}^{n}$. If $z, \zeta \in \mathbb{C}^{n}$, then $|z|=\left(z_{1} \bar{z}_{1}+\cdots+z_{n} \bar{z}_{n}\right)^{1 / 2}$, $\langle z, \zeta\rangle=z_{1} \zeta_{1}+\cdots+z_{n} \zeta_{n}$. The supporting function of a convex set $M$ in $\mathbb{C}^{n}$ is $H_{M}(\xi):=\sup _{z \in M} \operatorname{Re}\langle z, \xi\rangle$.

Let $\Omega$ be a convex bounded domain in $\mathbb{C}^{n}$ and $d(z):=\inf _{\zeta \in \partial \Omega}|z-\zeta|, z \in \Omega$. The space $A^{-\infty}(\Omega)$ of holomorphic functions in $\Omega$ with polynomial growth near the boundary $\partial \Omega$, equipped with its natural inductive limit topology, is defined as:

$$
A^{-\infty}(\Omega):=\left\{f \in \mathcal{O}(\Omega): \exists p>0, \sup _{z \in \Omega}|f(z)|[d(z)]^{p}<\infty\right\}
$$

Let $\mu$ be an analytic functional on $\mathbb{C}^{n}$, carried by a compact convex set $K$, and $(\mu * f)(z):=\left\langle\mu_{w}, f(z+w)\right\rangle$ be the corresponding convolution operator from $\mathcal{O}(\Omega+K)$ into $\mathcal{O}(\Omega)$. Denote by $\hat{\mu}$ the Fourier-Borel transformation of the analytic functional $\mu$, that is, $\hat{\mu}(\lambda)=\left\langle\mu_{z}, e^{\langle\lambda, z\rangle}\right\rangle, \lambda \in \mathbb{C}^{n}$.

[^0]
### 1.2. The main goal

The main goal of this Note is to establish surjectivity criteria for convolution operator $\mu *: A^{-\infty}(\Omega+K) \rightarrow A^{-\infty}(\Omega)$. It should be noted that the surjectivity of convolution operators for the spaces $\mathcal{O}(\Omega)$ of holomorphic functions in convex domains of $\mathbb{C}^{n}$ have been understood quite well (see, e.g., $[8,7,10]$ and references therein), whereas it is known less for the spaces of holomorphic functions with prescribed growth near the boundary of $\Omega$ (see [9]). Moreover, for the space $A^{-\infty}(\Omega)$, so far as we know, this problem is not yet treated, and therefore, there is much more to be investigated.

### 1.3. The main result

In [6] the so-called condition $(S)$ was introduced to study the convolution equations for hyperfunctions. Later, in [7], the surjectivity criterion for the convolution operators in holomorphic frame-work was established as the regular growth condition. By using argument of [7], in [4], it was obtained, for tube domains, that the surjectivity criterion is just condition (S), and finally, in [5], it was proved that condition $(S)$ is nothing but the regular growth condition.

For the case $A^{-\infty}$ we now introduce the following condition, which we call the condition ( $S^{a}$ ), stronger than (S).
Definition 1.1. An entire function $\varphi \in \mathcal{O}\left(\mathbb{C}^{n}\right)$ of exponential type is said to satisfy the condition $\left(S^{a}\right)$, if

$$
\begin{aligned}
& \exists s, N>0 \forall \zeta \in \mathbb{C}^{n}, \quad|\zeta|>N, \quad \exists \zeta^{\prime} \in \mathbb{C}^{n}, \quad\left|\zeta^{\prime}-\zeta\right|<\log (1+|\zeta|): \\
& \log \left|\varphi\left(\zeta^{\prime}\right)\right| \geqslant|\zeta| \cdot h_{\varphi}^{*}\left(\frac{\zeta}{|\zeta|}\right)-s \log |\zeta|
\end{aligned}
$$

where $h_{\varphi}^{*}(\zeta):=\lim \sup _{\zeta^{\prime} \rightarrow \zeta} \lim \sup _{r \rightarrow \infty} \frac{\log \left|\varphi\left(r \zeta^{\prime}\right)\right|}{r}$, the regularized radial indicator of $\varphi$.

Our main result is the following criterion for the convolution operator to be surjective on the class of all convex bounded domains in $\mathbb{C}^{n}$.

Theorem 1.2. Let $\mu$ be an analytic functional on $\mathbb{C}^{n}$, carried by a compact convex set $K$, and $\mu * A^{-\infty}(\Omega+K) \subseteq A^{-\infty}(\Omega)$ for any convex bounded domain $\Omega \subset \mathbb{C}^{n}$. The convolution operator $\mu *: A^{-\infty}(\Omega+K) \rightarrow A^{-\infty}(\Omega)$ is surjective for every $\Omega$ if and only if the regularized radial indicator of $\hat{\mu}$ coincides with $H_{K}$ and $\hat{\mu}$ satisfies $\left(S^{a}\right)$.

Corollary 1.3. Let $K$ be a singleton and either $n=1$, or $n>1$ and $\Omega$ has $C^{2}$ boundary. Each convolution operator acting from $A^{-\infty}(\Omega+K)$ into $A^{-\infty}(\Omega)$ is, in fact, a differential-difference operator of finite order generated by some polynomial and vice versa. Moreover, all nontrivial operators of such a type are surjective.

Remark that for $K$ being not a singleton the problem of surjectivity for a fixed (bounded convex) domain $\Omega$ is of great interest, and it still calls for investigation.

## 2. On the proof for Theorem 1.2

The proof of Theorem 1.2 is based on [1, Theorem 2.1] (see also [2,3] for more details), giving the description of the strong dual for $A^{-\infty}(\Omega)$ via Fourier-Borel transformation as the (FS)-space

$$
A_{\Omega}^{-\infty}=\left\{f \in \mathcal{O}\left(\mathbb{C}^{n}\right):|f|_{p}=\sup _{\zeta \in \mathbb{C}^{n}} \frac{|f(\zeta)|(1+|\zeta|)^{p}}{e^{H_{\Omega}(\zeta)}}<\infty, \text { for all } p \in \mathbb{N}\right\}
$$

and the following auxiliary results some of which have their own significance. Let

$$
A_{K}^{+\infty}:=\left\{g \in \mathcal{O}\left(\mathbb{C}^{n}\right): \sup _{\zeta \in \mathbb{C}^{n}} \frac{|g(\zeta)|}{(1+|\zeta|)^{p} e^{H_{K}(\zeta)}}<\infty, \text { for some } p \in \mathbb{N}\right\}
$$

## Lemma 2.1.

(1) If $\mu * A^{-\infty}(\Omega+K) \subseteq A^{-\infty}(\Omega)$, then $\hat{\mu} \in A_{K}^{+\infty}$.
(2) Conversely, if either $n=1$, or $n>1$ and both $\Omega+K, \Omega$ have $C^{2}$ boundary, and $\hat{\mu} \in A_{K}^{+\infty}$, then $\mu * A^{-\infty}(\Omega+K) \subseteq A^{-\infty}(\Omega)$.

Given a bounded convex domain $\Omega$ and a convex compact set $K$ in $\mathbb{C}^{n}$. We call an entire function $\varphi \in \mathcal{O}\left(\mathbb{C}^{n}\right)$ that satisfies $\varphi f \in A_{\Omega+K}^{-\infty}, \forall f \in A_{\Omega}^{-\infty}$, a multiplicator from $A_{\Omega}^{-\infty}$ into $A_{\Omega+K}^{-\infty}$. The set of all multiplicators from $A_{\Omega}^{-\infty}$ into $A_{\Omega+K}^{-\infty}$ is denoted by $\mathcal{M}_{\Omega, \Omega+K}^{-\infty}$. Each $\varphi \in \mathcal{M}_{\Omega, \Omega+K}^{-\infty}$ generates the multiplication operator $\Lambda_{\varphi}: f \in A_{\Omega}^{-\infty} \mapsto \varphi f \in A_{\Omega+K}^{-\infty}$.

Lemma 2.2. $\mathcal{M}_{\Omega, \Omega+K}^{-\infty}=A_{K}^{+\infty}$.

The multiplicator $\varphi \in A_{K}^{+\infty}$ is said to be a divisor from $A_{\Omega+K}^{-\infty}$ into $A_{\Omega}^{-\infty}$, if $f \in A_{\Omega+K}^{-\infty}$ and $\frac{f}{\varphi} \in \mathcal{O}\left(\mathbb{C}^{n}\right)$ imply $\frac{f}{\varphi} \in A_{\Omega}^{-\infty}$. Denote by $\mathcal{D}_{\Omega+K, \Omega}^{-\infty}$ the set of all divisors from $A_{\Omega+K}^{-\infty}$ into $A_{\Omega}^{-\infty}$.

Lemma 2.3. Let $\varphi \in A_{K}^{+\infty}$. Consider the following assertions:
(i) $\Lambda_{\varphi}\left(A_{\Omega}^{-\infty}\right)$ is closed in $A_{\Omega+K}^{-\infty}$;
(ii) For each $p \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C>0$ such that

$$
\sup _{\zeta \in \mathbb{C}^{n}} \frac{|f(\zeta)|(1+|\zeta|)^{p}}{e^{H_{\Omega}(\zeta)}} \leqslant C \sup _{\zeta \in \mathbb{C}^{n}} \frac{|\varphi(\zeta)||f(\zeta)|(1+|\zeta|)^{m}}{e^{H_{\Omega}(\zeta)+H_{K}(\zeta)}}, \quad \text { for all } f \in A_{\Omega}^{-\infty}
$$

(iii) $\varphi \in \mathcal{D}_{\Omega+K, \Omega}^{-\infty}$.

Then $(\mathrm{iii}) \Longrightarrow(\mathrm{i}) \Longleftrightarrow$ (ii).
Lemma 2.4. If $\varphi \in A_{K}^{+\infty}$ satisfies $\left(S^{a}\right)$, then $\varphi \in \mathcal{D}_{\Omega+K, \Omega}^{-\infty}$.
Lemma 2.5. Let $\varphi \in A_{K}^{+\infty}$. If the condition (ii) is valid for any bounded convex domain $\Omega$, then we have $h_{\varphi}^{*}(\zeta)=H_{K}(\zeta)$ and $\varphi$ satisfies ( $S^{a}$ ).

To prove Theorem 1.2 it is sufficient to do the following steps: use Lemmas 2.1 and 2.2 to connect the surjectivity operator with the multiplication one, then notice that the surjectivity of the convolution operator is equivalent to the condition (i) of Lemma 2.3, and finally apply Lemma 2.3 and Lemma 2.4 for the sufficiency, and together with Lemma 2.5 for necessity part of Theorem 1.2 , respectively.

## 3. Examples and explicit form for solutions

3.1. Examples of functions from $A_{K}^{+\infty}$ satisfying condition ( $S^{a}$ )

For $n=1$ the answer is always affirmative, as there obviously exists a function from $A_{K}^{+\infty}$ that satisfies condition $\left(S^{a}\right)$.
Proposition 3.1. For each convex compact set $K$ in $\mathbb{C}$ there exists a function $\varphi$ in $A_{K}^{+\infty}$ which satisfies $\left(S^{a}\right)$.
Corollary 3.2. Let $K=K_{1} \times \cdots \times K_{n}$, where $K_{j}$ are convex compact sets in $\mathbb{C}(1 \leqslant j \leqslant n)$. Then there exists a function $\varphi \in A_{K}^{+\infty}$ which satisfies $\left(S^{a}\right)$.

Example 3.3. Let $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N} \in \mathbb{C}^{n}$ and $P_{j}(\zeta) \in \mathbb{C}[\zeta](1 \leqslant j \leqslant N)$. Consider an exponential-polynomial $\varphi(\zeta):=$ $\sum_{j=1}^{N} P_{j}(\zeta) e^{\left\langle\lambda_{j}, \zeta\right\rangle}$. Set $\Lambda:=\left\{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N}\right\}$ and $K:=\operatorname{conv} \Lambda$ the convex hull of $\Lambda$. Then $h_{\varphi}^{*}=H_{K}$ and $\varphi$ satisfies $\left(S^{a}\right)$.

### 3.2. Explicit form for solutions

Assuming $n=1$, or $n>1$ and $\Omega, \Omega+K$ has $C^{2}$ boundary, we can prove the explicit representation of solutions:
Proposition 3.4. Let $\mu *$ be a surjective convolution operator from $A^{-\infty}(\Omega+K)$ onto $A^{-\infty}(\Omega)$. Then there exists a sequence $\Lambda=$ $\left(\lambda_{k}\right)_{k=1}^{\infty}$ in $\mathbb{C}^{n}$ with $\left|\lambda_{k}\right| \rightarrow \infty$ such that each function $g \in A^{-\infty}(\Omega)$ can be represented in the form

$$
\begin{equation*}
g(z)=\sum_{k=1}^{\infty} c_{k} e^{\left\langle\lambda_{k}, z\right\rangle}, \quad z \in \Omega \tag{3.1}
\end{equation*}
$$

and the function

$$
\begin{equation*}
f(w)=\sum_{k=1}^{\infty} \frac{c_{k}}{\hat{\mu}\left(\lambda_{k}\right)} e^{\left\langle\lambda_{k}, w\right\rangle}, \quad w \in \Omega+K \tag{3.2}
\end{equation*}
$$

belongs to $A^{-\infty}(\Omega+K)$ and is a solution of the equation $\mu * f=g$.

Proposition 3.5. Let $\mu$ be as in Theorem 1.2 and $\hat{\mu}$ satisfy $h_{\hat{\mu}}^{*}=H_{K}$ and $\left(S^{a}\right)$. Then there exists a sequence $\Lambda=\left(\lambda_{k}\right)_{k=1}^{\infty}$ in $\mathbb{C}^{n}$ with $\left|\lambda_{k}\right| \rightarrow \infty$ such that, for any $\Omega$, each function $g \in A^{-\infty}(\Omega)$ can be represented in the form (3.1) and the function $f$ as in (3.2) belongs to $A^{-\infty}(\Omega+K)$ and is a solution of the equation $\mu * f=g$.

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