C. R. Acad. Sci. Paris, Ser. I 348 (2010) 559–564

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Functional Analysis/Mathematical Physics

# Asymptotic velocity for fibered Hamiltonians

## *Sur la vitesse asymptotique des hamiltoniens fibrés*

Mondher Damak [a], Andrei Iftimovici [b]

[a] *University of Sfax, B.P. 802, 3018 Sfax, Tunisia*
[b] *University of Cergy-Pontoise, Department of Mathematics, CNRS, UMR 8088, 95000 Cergy-Pontoise, France*

ARTICLE INFO

*Article history:*
Received 11 February 2010
Accepted after revision 12 March 2010
Available online 31 March 2010

Presented by Gilles Pisier

ABSTRACT

We show the existence of the asymptotic velocity for a large class of Hamiltonians with a fibered structure.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

On prouve l'existence de la vitesse asymptotique pour une large classe d'hamiltoniens fibrés.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

### Version française abrégée

L'importance de la notion de vitesse asymptotique a été mise en évidence dans le contexte de la complétude asymptotique des systèmes quantiques à $N$ corps par J. Dereziński et C. Gérard (voir [3], le livre [4] et sa bibliographie). Pour clarifier et justifier nos développements ultérieurs, considérons une particule quantique dans l'espace physique $X=\mathbb{R}^d$ et ayant une structure interne dont l'espace des états est un espace de Hilbert complexe $\mathbf{E}$ (dans le cas le plus simple, celui d'une particule à spin, $\mathbf{E}$ est de dimension finie). Alors l'espace des états du système est $\mathscr{H}=L^2(X,\mathbf{E})$ et l'observable position est représentée par l'opérateur vectoriel $Q=(Q_1,\ldots,Q_d)$ où $Q_j$ est l'opérateur de multiplication dans $\mathscr{H}$ par la $j$-ème coordonnée. Si $H$ est l'hamiltonien du système alors $Q(t)=\mathrm{e}^{itH}Q\,\mathrm{e}^{-itH}$ représente l'observable position au moment $t$. Nous sommes naturellement intéressés par le comportement de $Q(t)/t$ pour $t\to\pm\infty$. Si, lorsque $|t|\to\infty$, cette expression converge (dans un sens à préciser) vers un opérateur vectoriel $V$, on dira que $V$ est la vitesse asymptotique du système. Le cas le plus élémentaire est celui où l'hamiltonien $H$ consiste seulement d'une partie cinétique *scalaire*, donc $H=h(P)$ où $P=-i\nabla$ est l'observable moment et $h:X\to\mathbb{R}$ est une fonction de classe $C^1$. Alors $Q(t)=Q+t\nabla h(P)$ et il est intuitivement clair que $Q(t)/t\to\nabla h(P)$. En fait, on peut montrer (voir [1]) que lorsque $|t|\to\infty$, on a $\mathrm{e}^{itH}\varphi(Q/t)\mathrm{e}^{-itH}\to\varphi(\nabla h(P))$ fortement pour toute fonction $\varphi\in C_{\mathrm{b}}(X)$ (complexe, continue, bornée).

Cependant ce type d'argument ne s'étend pas au cas où $h$ n'est plus une fonction scalaire, pour des motifs qui sont expliqués dans la Section 2. Or cette situation est physiquement incontournable puisqu'elle apparaît déjà dans les modèles de Pauli ou Dirac. Dans cet article nous étudions la vitesse asymptotique pour une classe générale d'hamiltoniens ayant une structure fibrée, classe d'opérateurs dont l'importance a été révélée par C. Gérard et F. Nier [5].

Avant d'énoncer le résultat central de l'article précisons une notation. Si $f$ est une fonction qui à presque tout $p$ dans $X$ associe un opérateur auto-adjoint $f(p)$ sur $\mathbf{E}$ et si l'application $p\mapsto(f(p)+i)^{-1}\in B(\mathbf{E})$ est fortement mesurable, alors

*E-mail addresses:* Mondher.Damak@fss.rnu.tn (M. Damak), Andrei.Iftimovici@u-cergy.fr (A. Iftimovici).

1631-073X/$ – see front matter © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2010.03.010

il existe un unique opérateur auto-adjoint $f(P)$ sur $\mathscr{H}$ tel que $\mathcal{F}(f(P)+i)^{-1}\mathcal{F}^{-1}$ soit l'opérateur de multiplication par la fonction $(f+i)^{-1}$ (ici $\mathcal{F}$ est la transformée de Fourier).

Le résultat central de l'article (le Théorème 2.1) peut être résumé ainsi :

*Supposons que pour chaque $p \in X$ on se donne un opérateur auto-adjoint $h(p)$ sur $\mathbf{E}$ de sorte que :*

(i) *l'application $p \mapsto (h(p)+i)^{-1} \in B(\mathbf{E})$ est fortement différentiable p.p. sur $X$,*
(ii) *pour presque tout $p \in X$ le spectre $\boldsymbol{\sigma}(h(p))$ de $h(p)$ est un ensemble discret.*

*Pour $j = 1, \ldots, d$ et pour presque tout $p \in X$ définissons un opérateur auto-adjoint $v_j(p)$ sur $\mathbf{E}$ par*

$$v_j(p) = \sum_{\lambda \in \boldsymbol{\sigma}(h(p))} E_{h(p)}\big(\{\lambda\}\big)\partial_j h(p) E_{h(p)}\big(\{\lambda\}\big).$$

*Les opérateurs $V_j = v_j(P)$ commutent deux à deux. Soit $V = (V_1, \ldots, V_d)$ et soit $E_V$ la mesure spectrale sur $X$ qui lui est associée. Alors pour toute application borelienne bornée $\varphi : \mathbb{R}^d \to \mathbb{C}$ telle que l'ensemble de ses points de discontinuité est de $E_V$-mesure nulle, on a* $\text{s-lim}_{|t|\to\infty} \mathrm{e}^{itH}\varphi(Q/t)\mathrm{e}^{-itH} = \varphi(V)$.

Pour plus de détails concernant la mesure $E_V$ voir la Section 1. Précisons que la définition de $\partial_j h(p)$ n'est pas triviale dans la généralité de l'énoncé, à moins que $\mathbf{E}$ ne soit de dimension finie. Cette question sera discutée en détail dans la Section 1 qui est consacrée à quelques développements nécessaires à la preuve du résultat principal, laquelle sera présentée dans la Section 2.

Les systèmes quantiques pour lesquels on peut calculer explicitement la vitesse asymptotique par la formule précédente sont nombreux. Par exemple, pour l'hamiltonien de Dirac on trouve $V = P(P^2+m^2)^{-1/2}$ (le cas plus général des hamiltoniens localement scalaires est similaire). Notons que les techniques introduites dans la Section 2 permettent de traiter des hamiltoniens qui ne deviennent fibrés qu'après une transformation unitaire ce qui permet de considérer le cas des particules en champs magnétiques et électriques et parfois d'obtenir même des formules explicites de $V$.

## 1. Preliminaries

We begin with some preliminary comments on vector-operators. Let $\mathscr{H}$ be a (complex) Hilbert space and let us fix an integer $d \geqslant 1$. We set $X = \mathbb{R}^d$, denote $q_j : X \to \mathbb{R}$ the $j$-th coordinate map and $\mathscr{B}(X)$ the algebra of bounded Borel functions $\varphi : X \to \mathbb{C}$. The spectral measure of a self-adjoint operator $S$ will be denoted $E_S$.

We call *vector-operator* a family $A = (A_1, \ldots, A_d)$ consisting of pairwise commuting self-adjoint operators on $\mathscr{H}$. The following fact is known: there is a unique Borel spectral measure $E_A$ on $X$ such that $E_A(q_j^{-1}(M)) = E_{A_j}(M)$ for each real Borel set $M$. This allows one to associate to any $\varphi \in \mathscr{B}(X)$ a bounded normal operator $\varphi(A)$ by setting $\varphi(A) = \int_X \varphi(\lambda) E_A(\mathrm{d}\lambda)$. Note that for $k \in X$ the operator $k_1A_1 + \cdots + k_dA_d$ is essentially self-adjoint and if we denote $kA$ its closure then $\mathrm{e}^{ikA} = \mathrm{e}^{ik_1A_1} \ldots \mathrm{e}^{ik_dA_d}$.

It is useful (see Proposition 1.1) to keep in mind the following basis independent description of vector-operators. Actually, there is a bijective correspondence between the following objects: (1) vector operators as defined above, (2) strongly continuous unitary representations $\{U_k\}_{k\in X}$ of $X$ on $\mathscr{H}$, (3) non-degenerate $*$-homomorphisms $C_0(X) \to B(\mathscr{H})$, and (4) unital and strictly continuous $*$-homomorphisms $C_\mathrm{b}(X) \to B(\mathscr{H})$. Here $C_\mathrm{b}(X)$ is the set of bounded continuous complex functions on $X$ and $C_0(X)$ the subspace of functions convergent to zero at infinity. For example, if $\{U_k\}_{k\in X}$ is given then $kA$ is *defined* as the unique self-adjoint operator such that $U_{tk} = \mathrm{e}^{itkA}$ for all $t \in \mathbb{R}$.

We say that $\varphi : X \to \mathbb{C}$ is $A$-almost everywhere ($A$-a.e.) continuous if the set of its discontinuity points has $E_A$ measure zero.

**Proposition 1.1.** *If $\{A(n)\}$ is a sequence of vector-operators then the following assertions are equivalent*:

(i) *There is a vector-operator $A$ such that* $\text{w-lim}_n \varphi(A(n)) = \varphi(A)$ *for all $\varphi \in C_0(X)$.*
(ii) $\text{s-lim}_n \mathrm{e}^{ikA(n)}$ *exists for any $k \in X$.*
(iii) $\text{s-lim}_n \varphi(A(n))$ *exists for any $\varphi \in C_\mathrm{b}(X)$.*

*Under these conditions $A$ is uniquely defined and* $\text{s-lim}_n \varphi(A(n)) = \varphi(A)$ *if $\varphi \in \mathscr{B}(X)$ is $A$-a.e. continuous.*

If the conditions of the preceding proposition are satisfied we say that the sequence $\{A(n)\}$ of vector-operators converges to $A$ and we write $A = \lim_n A(n)$.

Proposition 1.1 is a simple consequence of a standard fact concerning Borel probability measures (see [2]): if $\mu_n$ are probability measures on $X$ then there is a probability measure $\mu$ on $X$ such that $\lim_n \int \varphi \,\mathrm{d}\mu_n = \int \varphi \,\mathrm{d}\mu$ for all $\varphi \in C_0(X)$ if and only if there is a measure $\mu$ on $X$ such that $\lim_n \int \varphi \,\mathrm{d}\mu_n = \int \varphi \,\mathrm{d}\mu$ if $\varphi \in \mathscr{B}(X)$ is $\mu$-a.e. continuous and also if and only if the Fourier transforms of the $\mu_n$ converge on $X$ and their limit function is continuous at zero.

One more point has to be mentioned for the proof of (ii) $\Rightarrow$ (i). If we denote $U_k = \text{s-lim}_n e^{ikA(n)}$ then we clearly get a (strongly) Borel unitary representation of $X$ on $\mathscr{H}$. But note that $\mathscr{H}$ may be assumed separable (the cyclic subspace generated by the operators $A(n)$ and some fixed vector in $\mathscr{H}$ is separable) hence the map $k \mapsto U_k$ is strongly continuous. Then from the ($d$-dimensional version of the) Stone theorem it follows that there is a vector-operator $A$ such that $U_k = e^{ikA}$ and the rest of the proof is straightforward.

In the rest of this section we discuss some technical points concerning the derivatives of unbounded operator valued functions. This will allow us to give a precise meaning to the expression (2.1).

We fix a Hilbert space $\mathbf{E}$ and denote $B(\mathbf{E})$ the space of bounded operators on it. Let $S(\mathbf{E})$ be the set of self-adjoint *not necessarily bounded* operators on $\mathbf{E}$.

**Definition 1.2.** Let $\chi : I \mapsto S(\mathbf{E})$, where $I$ is a real open set. We say that $\chi$ is strongly differentiable at some point $s_0 \in I$ if there is $z \in \mathbb{C} \setminus \mathbb{R}$ such that the map $I \ni s \mapsto r_z(s) \equiv (z - \chi(s))^{-1} \in B(\mathbf{E})$ is strongly differentiable at $s_0$. In this case, we denote by $r'_z(s_0)$ the derivative of the map $r_z$ at $s_0$ and if this derivative exists at all $s \in I$ then we denote by $r'_z$ the map $I \ni s \mapsto r'_z(s) \in B(\mathbf{E})$.

Of course, one can similarly define the weak and norm differentiability but in the present paper we shall talk about "differentiability" of operator valued functions only in the sense of the preceding definition. Also, strong and norm continuity of a map $\chi : I \mapsto S(\mathbf{E})$ is interpreted in the resolvent sense. The first resolvent identity gives:

**Lemma 1.3.** *The definition of the differentiability of $\chi$ on $I$ does not depend on the choice of $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, for $z, \zeta \in \mathbb{C} \setminus \mathbb{R}$, we have $r'_\zeta = (1 + (z - \zeta) r_\zeta) r'_z (1 + (z - \zeta) r_\zeta)$.*

In particular we get $\|r'_\zeta(s)\| \leqslant \|r'_z(s)\|(1 + |z - \zeta| \|r_\zeta(s)\|)^2 \leqslant C(1 + |z - \zeta|/\Im\mathrm{m}\, \zeta)^2$. This estimate and the Helffer–Sjöstrand formula (see [6] or the proof of Lemma 1.6 below) yield:

**Lemma 1.4.** *If $\chi$ is differentiable on the open set $I \subset \mathbb{R}$ and if $\varphi \in C_c^\infty(\mathbb{R})$ then $I \ni s \mapsto \varphi(\chi(s)) \in B(\mathbf{E})$ is a differentiable map.*

We shall now give a meaning to the derivative $\chi'(s)$ as (unbounded) operator.

**Definition 1.5.** If $\chi : I \mapsto S(\mathbf{E})$ is differentiable at some real point $s_0$, we define the operator $\chi'(s_0)$ as a sesquilinear form on the domain of $\chi(s_0)$ by the formula

$$\chi'(s_0) = \big(z - \chi(s_0)\big) r'_z(s_0) \big(z - \chi(s_0)\big), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{1.1}$$

and we call it *derivative at $s_0$ of the (unbounded) self-adjoint operator valued map $\chi$*. If $\chi$ is differentiable on an open set $I \subset \mathbb{R}$ we denote by $\chi'$ the (unbounded) operator valued map $I \ni s \mapsto \chi'(s)$.

The definition does not depend on the choice of $z$. For, inserting $r_z(z - \chi)$ on the left and right of $r'_z$ given by Lemma 1.3 and by the resolvent identity $r_\zeta = (1 + (z - \zeta) r_\zeta) r_z$, we get $r'_\zeta = r_\zeta (z - \chi) r'_z (z - \chi) r_\zeta$.

For a self-adjoint operator $A : \mathrm{dom}(A) \subseteq \mathbf{E} \to \mathbf{E}$ we equip its domain $\mathrm{dom}(A)$ with the graph topology and we identify $\mathrm{dom}(A) \subseteq \mathbf{E} \subseteq \mathrm{dom}(A)^*$. Below we shall identify, for each $s \in I$, the continuous sesquilinear form $\chi'(s)$ defined above with a continuous operator $\chi'(s) : \mathrm{dom}(\chi(s)) \to \mathrm{dom}(\chi(s))^*$.

**Lemma 1.6.** *If $\varphi, \psi \in C_c^\infty(\mathbb{R})$ and $\chi : I \mapsto S(\mathbf{E})$ is differentiable on $I \subset \mathbb{R}$ then the map $\varphi(\chi)\chi\psi(\chi)$ is differentiable on $I$ and the Leibnitz rule is valid: $[\varphi(\chi)\chi\psi(\chi)]' = [\varphi(\chi)]'\chi\psi(\chi) + \varphi(\chi)\chi'\psi(\chi) + \varphi(\chi)\chi[\psi(\chi)]'$.*

**Proof.** Obviously $\varphi(\chi)\chi'\psi(\chi)$ is a $B(\mathbf{E})$ valued function. Setting $\psi_1(x) = x\psi(x)$ we get $\psi_1 \in C_c^\infty(\mathbb{R})$, hence by Lemma 1.4 the map $\varphi(\chi)\chi\psi(\chi) = \varphi(\chi)\psi_1(\chi)$ is differentiable. Further, the Helffer–Sjöstrand formula (see [6]) writes as:

$$\varphi(A) = \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar z} \widetilde{\varphi}(z)(z - A)^{-1} \, \mathrm{d}\mu(z),$$

where $A \in S(\mathbf{E})$ and $\tilde\varphi \in C_c^\infty(\mathbb{C})$ is an almost-analytic extension of $\varphi$ satisfying $\widetilde{\varphi}|_{\mathbb{R}} = \varphi$ and $|\partial_{\bar z} \widetilde{\varphi}(z)| \leqslant C_n |\Im\mathrm{m}\, z|^n$ for $n \in \mathbb{N}$ and $\mathrm{d}\mu(z) = \mathrm{d}x\,\mathrm{d}y$ is the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$. Then, for each $s \in I$, $\varphi(\chi(s))\chi(s)\psi(\chi(s)) = \pi^{-2} \int_{\mathbb{C}^2} \partial_{\bar z} \widetilde{\varphi}(z) \partial_{\bar\zeta} \widetilde{\psi}(\zeta) r_z(s)\chi(s) r_\zeta(s) \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(\zeta)$. It suffices thus to compute in each $s \in I$ the derivative of the map $r_z \chi r_\zeta$. But $r_z \chi r_\zeta = -r_z + \zeta r_z r_\zeta$, hence $[r_z \chi r_\zeta]' = r'_z(\zeta r_\zeta - 1) + \zeta r_z r'_\zeta$. By (1.1),

$$[r_z \chi r_\zeta]' = r'_z\big(\zeta r_\zeta - (\zeta - \chi) r_\zeta\big) + \zeta r_z r_\zeta \chi' r_\zeta = r'_z \chi r_\zeta + r_z \chi' r_\zeta + r_z(\zeta r_\zeta - 1)\chi' r_\zeta = r'_z \chi r_\zeta + r_z \chi' r_\zeta + r_z \chi r'_\zeta.$$

By differentiating the previous integral and replacing it in the last three terms, the Leibnitz rule will then follow via the Helffer–Sjöstrand formula. □

## 2. The main results

Let $\mathscr{H} = L^2(X, \mathbf{E})$ and let $Q = (Q_1, \ldots, Q_d)$ be the position vector-operator in $\mathscr{H}$. So $Q_j$ is the operator of multiplication by $q_j$ and if $\varphi \in \mathscr{B}(X)$ then $\varphi(Q)$ is the operator of multiplication by $\varphi$ in $\mathscr{H}$. If $t \neq 0$ is a real number, then $\varphi(Q/t)$ has an obvious meaning. We are interested in the behavior as $|t| \to \infty$ of the operators $\mathrm{e}^{itH} \frac{Q_j}{t} \mathrm{e}^{-itH}$, $j = 1, \ldots, n$, for a certain class of self-adjoint operators $H$.

Let $\mathcal{F}$ be the Fourier transform, a unitary operator on $\mathscr{H}$. If $h : X \to S(\mathbf{E})$ is such that $p \mapsto (h(p) + i)^{-1} \in B(\mathbf{E})$ is strongly Borel, we define $h(P)$ as the unique self-adjoint operator on $\mathscr{H}$ such that $\mathcal{F}(h(P) + i)^{-1}\mathcal{F}^{-1}$ is the operator of multiplication by the function $p \mapsto (h(p) + i)^{-1}$. Thus, formally $\mathcal{F}h(P)\mathcal{F}^{-1}$ is the operator of multiplication by the function $h$. It follows also easily that if $\varphi \in \mathscr{B}(\mathbb{R})$ then $\mathcal{F}\varphi(h(P))\mathcal{F}^{-1}$ is the operator of multiplication by the $B(\mathbf{E})$-valued function $p \mapsto \varphi(h(p))$.

The following theorem is our main result. The vector-operator $V$ introduced in it is the *asymptotic velocity* of the quantum system with Hamiltonian $H$.

**Theorem 2.1.** *Let $H = h(P)$ where the function $h : X \to S(\mathbf{E})$ has the following properties*:

(i) *The map $p \mapsto (h(p) + i)^{-1} \in B(\mathbf{E})$ is strongly Borel and almost everywhere strongly differentiable.*
(ii) *For almost each $p \in X$ the spectrum of $h(p)$ is a discrete set.*

*For $1 \leqslant j \leqslant d$ let $v_j$ be the map $X \to S(\mathbf{E})$ defined almost everywhere by*

$$v_j(p) = \sum_{\lambda \in \boldsymbol{\sigma}(h(p))} E_{h(p)}\big(\{\lambda\}\big)\partial_j h(p) E_{h(p)}\big(\{\lambda\}\big). \tag{2.1}$$

*Then $V = v(P) = (v_1(P), \ldots, v_d(P))$ is a vector-operator on $\mathscr{H}$ and if $\varphi \in \mathscr{B}(X)$ is continuous $V$-a.e. we have*

$$\text{s-lim}_{|t|\to\infty} \mathrm{e}^{itH} \varphi\left(\frac{Q}{t}\right) \mathrm{e}^{-itH} = \varphi(V) \equiv \varphi\big(v(P)\big). \tag{2.2}$$

**Remark.** If $\mathbf{E}$ is separable and if $p \mapsto (h(p) + i)^{-1}$ is locally Lipschitz then the condition (i) is fulfilled.

Note that (2.1) may be written $v(p) = \sum_{\lambda \in \boldsymbol{\sigma}(h(p))} E_{h(p)}(\{\lambda\}) \nabla h(p) E_{h(p)}(\{\lambda\})$ as elements of $X \otimes S(\mathbf{E})$.

As explained in Section 1, $\partial_j h(p)$ is in general not more than a sesquilinear form on the domain of $h(p)$, but this suffices to make $E_{h(p)}(\{\lambda\})\partial_j h(p) E_{h(p)}(\{\lambda\})$ a bounded self-adjoint operator on $\mathbf{E}$ for a.e. $p \in X$. Since the $E_{h(p)}(\{\lambda\})$ are pairwise orthogonal, the sum in (2.1) makes sense and defines a self-adjoint operator on $\mathbf{E}$, not bounded in general, which commutes with $h(p)$. If the operators $h(p)$ have a form domain $\mathbf{F}$ independent of $p$ and if $h(p)$ considered as operators $\mathbf{F} \to \mathbf{F}^*$ are strongly differentiable almost everywhere, the derivatives $\partial_j h(p)$ are well defined in the usual sense for a.e. $p \in X$ and the interpretation of the r.h.s. of (2.1) is much simpler.

Thus each component $V_j = v_j(P)$ is a self-adjoint operator which commutes with $H$. The operators $V_j$ and $V_k$ commute for any $j, k$ but this fact is not obvious (even if $\mathbf{E}$ is finite dimensional) when one refers only to the formula (2.1). But this follows easily from the relation (2.2), since the components of $Q$ commute.

In the scalar case $\mathbf{E} = \mathbb{C}$ we have $V = \nabla h(P)$ hence Theorem 2.1 is just Theorem 7.C.1 from [1]. The case $\dim \mathbf{E} \geqslant 2$ is however of a quite different nature and it is not trivial even if in the simplest case $\mathbf{E} = \mathbb{C}^2$. The main difficulty comes from the fact that the operators $h(p_1)$ and $h(p_2)$ do not commute if $p_1 \neq p_2$ and cannot be diagonalized in a smooth way. Indeed, assume that $\dim \mathbf{E} = N < \infty$ and that $h : X \to S(\mathbf{E})$ is of class $C^\infty$. Then for each $p$, $h(p)$ decomposes as $\sum_{k=1}^N \lambda_k(p)\pi_k(p)$ where $\lambda_k : X \to \mathbb{R}$ and $\pi_k : X \to S(\mathbf{E})$ are projector valued maps and satisfy $\pi_k(p)\pi_l(p) = \delta_{kl}\pi_k(p)$ and $\sum_{k=1}^N \pi_k(p) = 1$ for $p \in X$. However, the functions $\lambda_k$ are not more than twice differentiable (they are not $C^2$ and the derivatives are not Hölder continuous in general) and the maps $\pi_k$ are not even continuous in general (see for example [7, Example 5.12]). Hence the expression (2.1) of the asymptotic velocity is rather subtle, and neither the formula nor its proof cannot be inferred in any simple way from the scalar case. To summarize, we emphasize that the main difficulty one has to overcome in the proof of Theorem 2.1 is the lack of differentiability of the eigenvectors of the operators $h(p)$.

One more question arises naturally, namely that of the regularity of the asymptotic velocity $p \mapsto v(p)$ in relation to that of $h$. Although this goes beyond the aims of this paper, let us notice that in the scalar case the answer is trivial: if $h$ is of class $C^n$ then $v$ is of class $C^{n-1}$. In the simplest non-scalar case $d = 1$ and $\mathbf{E} = \mathbb{C}^2$ we can prove that *$v$ is strongly continuous if $p \mapsto (h(p) + i)^{-1}$ is strongly continuously differentiable* and this is already rather surprising knowing that the spectral projectors are not more than Borel in general.

We sketch the proof of Theorem 2.1. By Proposition 1.1 it suffices to show that for each $k \in X$

$$\text{s-lim}_{s\to 0} \mathrm{e}^{ih(P)/s} \mathrm{e}^{isk\cdot Q} \mathrm{e}^{-ih(P)/s} = \mathrm{e}^{ik\cdot v(P)}.$$

Since

$$\text{s-lim}_{s\to 0} e^{isk\cdot Q} = 1 \quad \text{and} \quad e^{-isk\cdot Q} e^{ih(P)/s} e^{isk\cdot Q} = e^{ih(P+sk)/s},$$

it will be enough to prove that $e^{ih(P+sk)/s} e^{-ih(P)/s} \to e^{ik\cdot v(P)}$ as $s \to 0$. For this, it suffices to show

$$\lim_{s\to 0} \int_X \left\| \left( e^{ih(p+sk)/s} e^{-ih(p)/s} - e^{ik\cdot v(p)} \right) f(p) \right\|_{\mathbf{E}}^2 \mathrm{d}p = 0 \quad \text{for all } f \in C_c^\infty(X, \mathbf{E}).$$

By dominated convergence it remains to show that $\lim_{s\to 0} \|(e^{ih(p+sk)/s} e^{-ih(p)/s} - e^{ik\cdot v(p)})u\|_{\mathbf{E}} = 0$ for almost each $p \in X$. In conclusion, we are reduced to proving the following result:

**Theorem 2.2.** *Let $I$ be a real open interval containing zero. Let the map $\chi : I \to S(\mathbf{E})$ be strongly differentiable at zero and such that the spectrum of $\chi(0)$ has no accumulation points. Denote $E_{\chi(0)}(\{\lambda\})$ by $\pi_\lambda(0)$. Then*

$$\text{s-lim}_{s\to 0} e^{i\chi(s)/s} e^{-i\chi(0)/s} = \exp\left( i \sum_{\lambda\in\boldsymbol{\sigma}(\chi(0))} \pi_\lambda(0)\chi'(0)\pi_\lambda(0) \right). \tag{2.3}$$

Note that for each $\lambda \in \boldsymbol{\sigma}(\chi(0))$, $\pi_\lambda(0)\chi'(0)\pi_\lambda(0) \in B(\mathbf{E})$ and obviously this operator commutes with $\chi(0)$.

For the proof of Theorem 2.2 we need some preparations. Let $C_\infty(\mathbb{R})$ denote the algebra of functions that have a limit at infinity. Recall that if $\chi : I \to S(\mathbf{E})$ is norm (respectively strongly) continuous in zero, then for each $\varphi \in C_\infty(\mathbb{R})$ (respectively $\varphi \in C_b(\mathbb{R})$) we have $\varphi(\chi(s)) \to \varphi(\chi(0))$ as $s \to 0$ in norm (respectively strongly) in $B(\mathbf{E})$. We recall also that strong differentiability implies norm continuity.

**Lemma 2.3.** *If $\chi$ is norm continuous in zero and if $\lambda \in \boldsymbol{\sigma}(\chi(0))$ is an isolated eigenvalue then for $|s|$ small enough there are $\alpha > 0$ and $\beta > 1$ such that $\boldsymbol{\sigma}(\chi(s)) \cap ([\lambda - \alpha/\beta, \lambda - \alpha/\beta^2] \cup [\lambda + \alpha/\beta^2, \lambda + \alpha/\beta]) = \emptyset$. Moreover, if $I = (\lambda - \alpha/\beta^2, \lambda + \alpha/\beta^2)$, then $\|E_{\chi(s)}(I) - E_{\chi(0)}(I)\| \to 0$ as $s \to 0$.*

**Proof.** There is some $\alpha > 0$ such that $(\lambda - \alpha, \lambda + \alpha) \cap \boldsymbol{\sigma}(\chi(0)) = \{\lambda\}$. Let then $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi \leqslant 1$, such that $\varphi = 1$ on $[\lambda - \alpha/\beta, \lambda - \alpha/\beta^2] \cup [\lambda + \alpha/\beta^2, \lambda + \alpha/\beta]$ and $\varphi = 0$ on $(-\infty, \lambda - \alpha] \cup [\lambda - \alpha/\beta^3, \lambda + \alpha/\beta^3] \cup [\lambda + \alpha, \infty)$. Thus $\varphi(\chi(0)) = 0$ and since $\lim_{s\to 0} \|\varphi(\chi(s)) - \varphi(\chi(0))\| = 0$ we get $\|\varphi(\chi(s))\| = \sup_{\mu\in\boldsymbol{\sigma}(\chi(s))} |\varphi(\mu)| < 1/2$ for $|s|$ small enough. This implies the first assertion. For the second one, we choose a $\phi \in C_c^\infty(\mathbb{R})$, $\phi \leqslant 1$, such that $\phi = 1$ on $I$ and $\phi = 0$ on $(-\infty, \lambda - \alpha/\beta] \cup [\lambda + \alpha/\beta, \infty)$. The first assertion of the lemma ensures $\phi(\chi(s)) = E_{\chi(s)}(I)$ for $|s|$ small enough thus we may conclude by the norm continuity of $\chi$ in zero. □

Now we fix an isolated eigenvalue $\lambda$ of $\chi(0)$. Let $\delta \in [\alpha/\beta^2, \alpha/\beta]$ and let $J \equiv [\lambda - \delta, \lambda + \delta]$. We denote $\pi_\lambda(s) \equiv E_{\chi(s)}(J)$ (which is coherent with $\pi_\lambda(0) = E_{\chi(0)}(\{\lambda\})$). By the proof of the lemma, if $|s|$ is small enough, $\pi_\lambda(s) = \phi(\chi(s))$ and since $\phi$ is smooth, the spectral projector has the same regularity as the map $\chi$ (see Theorem 6.2.5 in [1]). On the other hand, for such $s$ the formula

$$\pi_\lambda(s) = \frac{1}{2\pi i} \oint_{|z-\lambda|=\delta} \frac{\mathrm{d}z}{z - \chi(s)}$$

provides, via the results from Section 1, another way to compute explicit expressions for the derivatives of the map $s \mapsto \pi_\lambda(s)$.

**Lemma 2.4.** *Let $\chi$ be strongly differentiable at zero and suppose that $\boldsymbol{\sigma}(\chi(0))$ has no accumulation points. If $\lambda \in \boldsymbol{\sigma}(\chi(0))$, the projector valued map $\pi_\lambda$ is strongly differentiable at zero and*

$$\pi'_\lambda(0) = 2\mathfrak{Re} \sum_{\mu\neq\lambda} \frac{\pi_\lambda(0)\chi'(0)\pi_\mu(0)}{\lambda - \mu}.$$

*Moreover, $\sum_\lambda \pi'_\lambda(0) = 0$.*

**Proof of Theorem 2.2.** Since the spectrum of $\chi(0)$ is pure point, it is sufficient to prove (2.3) on an eigenvector $u$ corresponding to a (fixed) eigenvalue $\lambda$ of $\chi(0)$, so we are reduced to look at $\text{s-lim}_{s\to 0} e^{i(\chi(s)-\lambda)/s}\pi_\lambda(0)$. We shall consider only $s$ small enough and use the objects introduced after Lemma 2.3. Then, by this lemma, $\|(1 - \pi_\lambda(s))e^{i(\chi(s)-\lambda)/s}\pi_\lambda(0)\| = \|(1 - \pi_\lambda(s))\pi_\lambda(0)\| \to \|(1 - \pi_\lambda(0))\pi_\lambda(0)\| = 0$, as $s \to 0$. We thus have to look at $\text{s-lim}_{s\to 0} e^{i(\chi(s)-\lambda)/s}\pi_\lambda(s)\pi_\lambda(0)$. Using again Lemma 2.3, it remains to compute

$$\text{s-lim}_{s\to 0} e^{i(\chi(s)-\lambda)/s}\pi_\lambda(s) = \text{s-lim}_{s\to 0} e^{i(\chi(s)\pi_\lambda(s)-\lambda\pi_\lambda(s))/s}\pi_\lambda(s).$$

Since $\chi(s)\pi_\lambda(s) \in B(\mathbf{E})$, we can reduce to the convergence of the exponent. It suffices thus to prove the strong convergence of $(\chi(s)\pi_\lambda(s) - \lambda\pi_\lambda(s))/s = (\pi_\lambda(s)\chi(s)\pi_\lambda(s) - \pi_\lambda(0)\chi(0)\pi_\lambda(0))/s - \lambda(\pi_\lambda(s) - \pi_\lambda(0))/s$ as $s \to 0$ and to compute the limit. By Lemma 2.4 the second term in the r.h.s. converges to $-\lambda\pi'_\lambda(0)$. For the first one, we consider a function $\theta \in C_c^\infty(\mathbb{R})$ with support included in $[\lambda - \alpha/\beta, \lambda + \alpha/\beta]$ and such that $\theta = 1$ on $J$. Then $\pi_\lambda(s) = \theta(\chi(s))$, hence $\pi'_\lambda(0) = [\theta(\chi)]'(0)$ On the other hand, $(\pi_\lambda(s)\chi(s)\pi_\lambda(s) - \pi_\lambda(0)\chi(0)\pi_\lambda(0))/s$ can be seen as $(\vartheta(s) - \vartheta(0))/s$, where $\vartheta = \widetilde{\theta} \circ \chi$ and $\widetilde{\theta}(x) = \theta(x)x\theta(x)$. The map $\vartheta$ has the same regularity properties as $\chi$ and by Lemma 1.6 we get s-$\lim_{s\to 0}(\pi_\lambda(s)\chi(s)\pi_\lambda(s) - \pi_\lambda(0)\chi(0)\pi_\lambda(0))/s$ as $\pi_\lambda(0)\chi'(0)\pi_\lambda(0) + \pi'_\lambda(0)\chi(0)\pi_\lambda(0) + \pi_\lambda(0)\chi(0)\pi'_\lambda(0)$ which equals $\pi_\lambda(0)\chi'(0)\pi_\lambda(0) + \lambda\pi'_\lambda(0)$ since $\pi'_\lambda = \pi_\lambda\pi'_\lambda + \pi'_\lambda\pi_\lambda$. □

## Acknowledgements

The authors are indebted to Vladimir Georgescu for enlightening discussions.

## References

[1] W. Amrein, A. Boutet de Monvel, V. Georgescu, $C_0$-groups, Commutator Methods and Spectral Theory of $N$-body Hamiltonians, Birkhäuser Verlag, 1996.
[2] N. Bourbaki, Eléments de mathématique : Intégration, Ch. VII, Hermann, Paris, 1967.
[3] J. Dereziński, Algebraic approach to the $N$-body long range scattering, Rev. Math. Phys. 3 (1) (1991) 1–62.
[4] J. Dereziński, C. Gérard, Scattering Theory of Classical and Quantum $N$-particle Systems, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997, xii+444 pp.
[5] C. Gérard, F. Nier, The Mourre theory for analytically fibered operators, J. Funct. Anal. 152 (1) (1998) 202–219.
[6] B. Helffer, J. Sjöstrand, Equation de Schrödinger avec champ magnétique et équation de Harper, in: Schrödinger Operators, Sonderborg, 1988, in: Lecture Notes in Phys., vol. 345, Springer, Berlin, 1989, pp. 118–197.
[7] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1966.