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Partial Differential Equations

Existence of solutions for semilinear elliptic problems in exterior of ball

Existence des solutions pour des problèmes elliptiques non linéaires à extérieur de la boule

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ABSTRACT

We prove the existence of solutions for the semilinear elliptic problem in $\Omega = B(0,R)^c$, $N \geqslant 3$.

$$-\Delta u = G'(u),$$

under suitable general assumptions on the nonlinear term G.

RÉSUMÉ

Dans cette Note, nous demontrons l'existence d'une solution pour des équation elliptiques non linéaires in $\Omega = B(0,R)^c$, $N \geqslant 3$

$$-\Delta u = G'(u),$$

pour a general nonlinéarité G.

In this Note we consider the semilinear elliptic problem in the exterior of a ball, $N \geqslant 3$

$$\begin{cases} -\Delta u = G'(u) & \text{on } \Omega = B(0,R)^c, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $B(0,R)^c = \{x \in \mathbb{R}^N \text{ such that } |x| > R\}$ and $G = -\frac{1}{2}u^2 + R(u) \in C^2$ fulfill the hypotheses

$$\left|R'(u)\right| \leqslant c_1|u|^{p-1} + c_2|u|^{q-1} 2 < p \leqslant q < \frac{2N}{N-2}; \tag{2}$$

$$\text{there exists } \xi_0 > 0 \text{ s.t. } G(\xi_0) > 0. \tag{3}$$

Eq. (1) has been intensively studied in case $\Omega = \mathbb{R}^N$, see e.g. [3], and in case Ω bounded domain with regular boundary for a wide class of nonlinearities, see e.g. [1]. Eq. (1) is the Euler–Lagrange equation associated to the following functional $I : H_0^1(\Omega) \to \mathbb{R}$ given by

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$$I(u) = \frac{1}{2}\|u\|^2_{H^1_0(\Omega)} - \int R(u)\,dx. \tag{4}$$

It is well known that the functional $I(u)$ exhibits a mountain pass geometry (see Proposition 1) and in this scenario the classical deformation lemma asserts that a Palais–Smale sequence (PS) exists at critical level c. The first and crucial difficulty is to give an a priori estimate on the Palais–Smale sequence, i.e. to prove that u_n is bounded in $H^1_0(\Omega)$ in case of general nonlinearity not fulfilling the classical Ambrosetti–Rabinowitz condition.

In an abstract framework, given an Hilbert space H let we consider the family of functionals that shows a mountain pass geometry for $\lambda = 1$

$$I(\lambda, u) = \frac{1}{2}\|u\|^2_H - \lambda J(u), \tag{5}$$

where $J \in C^2(H, \mathbb{R})$ and $\lambda \in \mathbb{R}^+$ and $\nabla J : H \to H$ is a compact mapping.

Theorem 1.1 of [5] states that there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times H$ such that

$$\begin{cases} u_n \text{ is a critical point of } I(\lambda_n, u)\ \lambda_n \to 1, \\ I(\lambda_n, u_n) \text{ bounded.} \end{cases} \tag{6}$$

As a matter of fact the existence of a sequence u_n of solution for the approximated problem does not guarantee in general that we can pass to the limit and prove that a solution for the case $\lambda = 1$ exists. The main difficulty is again the a priori estimate on the approximated solutions u_n. In some cases, a Pohozaev type identity applied to the approximated problem, guarantees the boundness of the u_n sequence and then the existence of a solution for the original problem, see e.g. [4] and [2] in the case of nonlinear Schrödinger equation in R^N.

In this Note we show that a Pohozaev type identity for the perturbed equation exists and that this constraint gives the boundness of the perturbed solutions. Therefore, under the above mentioned hypotheses we have the following

Theorem 1 *(main theorem). If* (2), (3) *hold then functional* (4) *has a mountain pass critical point.*

In order to prove the main theorem we define the perturbed functional $I(\lambda, .) : H^1_r(\Omega) \to \mathbb{R}$

$$I(\lambda, u) = \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \lambda \int R(u)\,dx, \tag{7}$$

where the nonlinear term is weakly continuous in

$$H^1_r(\Omega) = \{u \in H^1_0(\Omega) \text{ such that } u \text{ radially symmetric}\}.$$

Before to prove the main theorem some preliminaries are in order:

Proposition 1. *If* (2), (3) *hold then functional* (4) *has a mountain pass geometry.*

Proof. We notice simply that

$$I(u) \geqslant \frac{1}{2}\|u_n\|^2_{H^1_r(\Omega)} - c_1\|u_n\|^p_{H^1_r(\Omega)} - c_2\|u_n\|^q_{H^1_r(\Omega)},$$

and that the sequence u_n defined as follows

$$u_n(r) = \begin{cases} \xi_0(|x| - R_n + 1) & \text{for } R_n - 1 \leqslant |x| \leqslant R_n, \\ \xi_0 & \text{for } R_n \leqslant |x| \leqslant 2R_n, \\ \xi_0(2R_n - |x| + 1) & \text{for } 2R_n \leqslant |y| \leqslant 2R_n + 1, \\ 0 & \text{for } |x| \geqslant 2R_n + 1, \end{cases}$$

where ξ_0 is defined in (3) fulfills $I(u_n) < 0$ for $R_n \to \infty$. Indeed

$$\int_\Omega |\nabla u_n|^2\,dx = O(R_n^{N-1})$$

and

$$\int_\Omega G(u_n)\,dx = \int_{R_n}^{2R_n} r^{N-1} G(\xi_0)\,dr + O(R_n^{N-1}) = CG(\xi_0)R_n^N + O(R_n^{N-1}).$$

Since $G(\xi_0) > 0$ it follows that $I(u_n)$ is negative for n large enough. □

We show now a Pohozaev type identity that is crucial for the a priori estimate.

Lemma 1. *Let u be a solution of*

$$-\Delta u + u = \lambda R'(u) \quad \text{on } \Omega = B(0,R)^c,$$

then we have

$$\frac{u'^2(R)R^N}{2} + \frac{2-N}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x = -\lambda N\int_\Omega R(u)\,\mathrm{d}x + \frac{N}{2}\int_\Omega |u|^2\,\mathrm{d}x.$$

Proof. We write (1) using the radial symmetry of the solution

$$-u'' - \frac{N-1}{r}u' + u = \lambda R'(u), \tag{8}$$

then we have

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u'^2}{2}r^{2N-2}\right) = \big(\lambda R'(u) - u\big)r^{2N-2}u'.$$

We get

$$-\int_R^\infty \frac{1}{r^{N-2}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u'^2}{2}r^{2N-2}\right)\mathrm{d}r = \lambda\int_R^\infty R'(u)r^N u'\,\mathrm{d}r - \int_R^\infty u r^N u'\,\mathrm{d}r,$$

and by integration by parts we have

$$\frac{u'^2(R)R^N}{2} + (2-N)\int_R^\infty \frac{u'^2}{2}r^{N-1}\,\mathrm{d}r = \lambda\int_R^\infty \frac{\mathrm{d}}{\mathrm{d}r}\big(R(u)\big)r^N\,\mathrm{d}r - \int_R^\infty \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2}|u|^2\right)r^N\,\mathrm{d}r,$$

and hence

$$\frac{u'^2(R)R^N}{2} + \frac{2-N}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x = -N\lambda\int_\Omega R(u)\,\mathrm{d}x + \frac{N}{2}\int_\Omega |u|^2\,\mathrm{d}x. \qquad \square$$

Lemma 2. *Let* $(\lambda_n, u_n) \in \mathbb{R} \times H^1_r(\Omega)$ *be a sequence such that* $\nabla I(\lambda_n, u_n) = 0$, $\lambda_n \to 1$ *and* $I(\lambda_n, u_n)$ *bounded. Then* $\|u_n\|_{H^1_r(\Omega)}$ *is bounded.*

Proof. *Step I*: $\|u_n\|_{D^{1,2}(\Omega)}$ is bounded. We have thanks to Lemma 1

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \big(|\nabla u_n|^2 + |u_n|^2\big)\,\mathrm{d}x - \lambda_n\int_\Omega R(u_n)\,\mathrm{d}x \leqslant K, \\ \dfrac{u_n'^2(R)R^N}{2N} + \dfrac{2-N}{2N}\displaystyle\int_\Omega |\nabla u_n|^2\,\mathrm{d}x = -\lambda_n\int_\Omega R(u_n)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |u_n|^2\,\mathrm{d}x. \end{cases} \tag{9}$$

By adding the equations we get

$$\frac{u_n'^2(R)R^N}{2N} + \frac{1}{N}\int_\Omega |\nabla u_n|^2\,\mathrm{d}x \leqslant K.$$

Step II: $\|u_n\|_{L^2(\Omega)}$ is bounded.

Thanks to (2) and the interpolation inequality we have

$$I(\lambda_n, u_n) \geqslant \frac{1}{2}\|u_n\|^2_{H^1_r(\Omega)} - c_1\lambda_n\|u_n\|^{\alpha_1 p}_{L^2(\Omega)}\|u_n\|^{(1-\alpha_1)p}_{L^{2^*}(\Omega)} - c_2\lambda_n\|u_n\|^{\alpha_2 q}_{L^2(\Omega)}\|u_n\|^{(1-\alpha_2)q}_{L^{2^*}(\Omega)}, \tag{10}$$

where $\alpha_1 = \frac{N}{p} - \frac{N-2}{2}$ and $\alpha_2 = \frac{N}{q} - \frac{N-2}{2}$.

The Sobolev inequality gives

$$I(\lambda_n, u_n) \geqslant \frac{1}{2}\|u_n\|^2_{H^1_r(\Omega)} - c_1\lambda_n\|u_n\|^{\alpha_1 p}_{L^2(\Omega)}\|u_n\|^{(1-\alpha_1)p}_{D^{1,2}(\Omega)} - c_2\lambda_n\|u_n\|^{\alpha_2 q}_{L^2(\Omega)}\|u_n\|^{(1-\alpha_2)q}_{D^{1,2}(\Omega)}. \tag{11}$$

The fact that $\alpha_1 p < 2$ and $\alpha_2 q < 2$ for any $p, q > 2$ proves the boundness of $\|u_n\|_{L^2(\Omega)}$. $\square$

Proof of the main theorem. Let u_n be a sequence such that $\nabla I(\lambda_n, u_n) = 0$, $\lambda_n \to 1$ and $I(\lambda_n, u_n)$ bounded. The existence of such sequence is proved in [5]. The a priori estimate for u_n is given by Lemma 2. There exist $\bar{u}$ such that $u_n \to \bar{u}$ a.e. and by Strauss theorem [6] we have up to subsequences $\|u_n - \bar{u}\|_{L^p(\Omega)} = o(1)$ for $2 < p < \frac{2N}{N-2}$. We have

$$-\Delta u_n + u_n - R'(u_n) = \lambda_n R'(u_n) - R'(u_n) = o(1) \quad \text{in } H^{-1}(\Omega), \tag{12}$$

and hence u_n is a Palais–Smale sequence for the functional I. Indeed by (2) we have

$$\left|\int_\Omega (\lambda_n - 1) R'(u_n)\varphi \, dx\right| \leqslant |\lambda_n - 1| \left(c_1 \int_\Omega |u_n|^{p-1}|\varphi| \, dx + c_2 \int_\Omega |u_n|^{q-1}|\varphi| \, dx\right), \tag{13}$$

and hence

$$\left|\int_\Omega (\lambda_n - 1) R'(u_n)\varphi \, dx\right| \leqslant |\lambda_n - 1| \left(c_1 \|u_n\|_{H^1(\Omega)}^{p-1} \|\varphi\|_{H^1(\Omega)} + c_2 \|u_n\|_{H^1(\Omega)}^{q-1} \|\varphi\|_{H^1(\Omega)}\right). \tag{14}$$

Let us consider two functions u_n and u_m in the PS sequence, by subtraction we get

$$-\Delta(u_n - u_m) + (u_n - u_m) - \left(R'(u_n) - R'(u_m)\right) \to 0, \tag{15}$$

and we obtain

$$\int_\Omega \left|\nabla(u_n - u_m)\right|^2 dx + \int_\Omega \left|(u_n - u_m)\right|^2 dx - \int_\Omega \left(R'(u_n) - R'(u_m)\right)(u_n - u_m) \, dx = o(1). \tag{16}$$

Indeed by (2) we have

$$\begin{aligned}\int_\Omega \left|\left(R'(u_n) - R'(u_m)\right)(u_n - u_m)\right| dx &\leqslant c_1 \left(\int_\Omega |u_n|^{p-1}|u_n - u_m| \, dx + \int_\Omega |u_m|^{p-1}|u_n - u_m| \, dx\right) \\ &\quad + c_2 \left(\int_\Omega |u_n|^{q-1}|u_n - u_m| \, dx + \int_\Omega |u_m|^{q-1}|u_n - u_m| \, dx\right) = o(1).\end{aligned} \tag{17}$$

Eventually we have

$$\int_\Omega \left|\nabla(u_n - u_m)\right|^2 dx + \int_\Omega \left|(u_n - u_m)\right|^2 dx \to 0, \tag{18}$$

i.e. u_n is a Cauchy sequence in $H_r^1(\Omega)$. We obtain $\|u_n - \bar{u}\|_{H_r^1(\Omega)} = o(1)$. □

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