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Mathematical Analysis

On some inequalities of Bourgain, Brezis, Maz'ya, and Shaposhnikova related to L^1 vector fields

Sur certaines inégalités de Bourgain, Brezis, Maz'ya et Shaposhnikova concernant les champs de vecteurs dans L^1

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ABSTRACT

Bourgain and Brezis established, for maps $f \in L^n(\mathbb{T}^n)$ with zero average, the existence of a solution $\boldsymbol{Y} \in W^{1,n} \cap L^\infty$ of (1) $\operatorname{div} \boldsymbol{Y} = f$. Maz'ya proved that if, in addition, $f \in H^{n/2-1}(\mathbb{T}^n)$, then (1) can be solved in $H^{n/2} \cap L^\infty$. Their arguments are quite different. We present an elementary property of fundamental solutions of the biharmonic operator in two dimensions. This property unifies, in two dimensions, the two approaches, and implies another (apparently unrelated) estimate of Maz'ya and Shaposhnikova. We discuss higher dimensional analogs of the above results.

RÉSUMÉ

Bourgain and Brezis ont montré que, si $f \in L^n(\mathbb{T}^n)$ est de moyenne nulle, alors (1) $\operatorname{div} \boldsymbol{Y} = f$ a une solution $\boldsymbol{Y} \in W^{1,n} \cap C^0$. Maz'ya a prouvé que si, de plus, on a $f \in H^{n/2-1}(\mathbb{T}^n)$, alors il existe une solution de (1) dans $H^{n/2} \cap L^\infty$. Les deux preuves sont distinctes. Dans cette note, nous présentons une propriété élémentaire des solutions fondamentales de l'opérateur biharmonique en dimension deux. Cette propriété unifie, en dimension deux, les approches de Bourgain–Brezis et Maz'ya, et implique une autre estimation de Maz'ya et Shaposhnikova (apparemment non liée aux précédentes). Nous discutons des variantes de ces résultats en dimension supérieure.

In their pioneering work [1], Bourgain and Brezis proved that, when a map $f \in L^n(\mathbb{T}^n)$ has zero average,

$$\operatorname{div} \boldsymbol{Y} = f \tag{1}$$

has a solution $\boldsymbol{Y} \in W^{1,n} \cap C^0$. By duality, this result is trivially equivalent to the estimate

$$\|u\|_{L^{n/(n-1)}} \lesssim \|\nabla u\|_{W^{-1,n'}+L^1}, \quad \forall u \in L^n(\mathbb{T}^n) \text{ such that } \int_{\mathbb{T}^n} u = 0. \tag{2}$$

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The proof of (1) is very elaborate (the construction of $\boldsymbol{Y}$ is explicit and based on a nonlinear mechanism). So far, there is no straightforward argument yielding (2) when $n \geqslant 3$. However, when $n = 2$, Bourgain and Brezis [1] present a direct proof of (2) which relies on Fourier series, and more specifically on the fact that

$$\sum_{m\in\mathbb{Z}^2\setminus\{0\}} \frac{m_1 m_2}{(m_1^2+m_2^2)^2} e^{\imath m\cdot x} \in L^\infty \quad \text{and} \quad \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \frac{m_1^2 - m_2^2}{(m_1^2+m_2^2)^2} e^{\imath m\cdot x} \in L^\infty. \tag{3}$$

Assertion (1) is equivalent to the fact that for every vector field $\boldsymbol{X} \in W^{1,n}(\mathbb{T}^n)$ there is some $\boldsymbol{Y} \in W^{1,n} \cap C^0$ such that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$, where $\operatorname{div} \boldsymbol{Z} = 0$. For $n \geqslant 3$, a more involved version of this result has been established by Bourgain and Brezis [2,3]: in the previous decomposition, one may pick $\boldsymbol{Z}$ such that $\operatorname{curl} \boldsymbol{Z} = 0$. This implies new regularity results for the Hodge decomposition [2,3]. For example, when $n = 3$, Bourgain and Brezis [3] prove, for vectors fields $\boldsymbol{f} \in L^3(\mathbb{T}^3)$ such that $\operatorname{div} \boldsymbol{f} = 0$ and $\int_{\mathbb{T}^3} \boldsymbol{f} = 0$, the existence of a $\boldsymbol{Y} \in W^{1,3} \cap C^0(\mathbb{T}^3)$ such that

$$\operatorname{curl} \boldsymbol{Y} = \boldsymbol{f}. \tag{4}$$

Maz'ya [4] studied the solvability of (1) when $f \in H^{n/2-1}(\mathbb{T}^n)$ has zero average. The main result there is the existence of a solution $\boldsymbol{Y} \in H^{n/2} \cap L^\infty$ of (1). The proof of Maz'ya [4] is by duality, based on the estimate

$$\|u\|_{H^{1-n/2}} \lesssim \|\nabla u\|_{H^{-n/2}+L^1}, \quad \forall u \in H^{1-n/2}(\mathbb{T}^n) \text{ such } \int_{\mathbb{T}^n} u = 0. \tag{5}$$

Actually, [4] contains a version of (5) in $\mathbb{R}^n$ instead of $\mathbb{T}^n$ and with sharp constants. The proof of (5) is based on explicit formulae for the Fourier transform of singular integral operators, in the spirit of Stein and Weiss [7], Chapter IV, Theorem 4.5, p. 164. In dimension two, (5) is the same as (2) and provides a third argument leading to the solvability of (1) in $H^1 \cap C^0(\mathbb{T}^2)$.

In a different direction, Maz'ya and Shaposhnikova [6] proved the following estimates: for $u \in C^\infty(\mathbb{T}^n)$, one has

$$\left|\int_{\mathbb{T}^n} \partial_1 u \partial_2 u\right| + \left|\int_{\mathbb{T}^n} \left((\partial_1 u)^2 - (\partial_2 u)^2\right)\right| \lesssim \left(\int \left|(-\Delta)^{n/4+1/2} u\right|\right)^2. \tag{6}$$

Their approach is again based on Fourier transform formulae for singular integral operators, in the spirit of the proofs of (5) in [4] and of the $H^{3/2}$-regularity result in [5], and apparently unrelated to the proof of (2) via (3) in [1].

Our first contribution is the following: we revisit and connect, in two dimensions, (3) and (6) using a partial differential equations viewpoint. More specifically, our starting point is the following

Proposition 1. *In $\mathbb{R}^2$, the operator Δ^2 has a fundamental solution F such that $\partial_1\partial_2 F$, $\partial_1^2 F - \partial_2^2 F \in L^\infty$.*

Proof. Let $F(x) = \frac{1}{8\pi}|x|^2 \ln|x| - \frac{1}{16\pi}|x|^2$. One checks easily that $\partial_j^2 F = \frac{1}{4\pi}\ln|x| + \frac{1}{4\pi}\frac{x_j^2}{|x|^2}$, $\partial_1\partial_2 F = \frac{1}{4\pi}\frac{x_1 x_2}{|x|^2}$. In particular, $\Delta F = \frac{1}{2\pi}\ln|x|$, so that $\Delta^2 F = \delta$, while $\partial_1\partial_2 F$, $\partial_1^2 F - \partial_2^2 F \in L^\infty$. □

Corollary 2. *Let G be the (unique modulo constants) solution of $\Delta^2 G = \delta - (1/2\pi)^2$ on $\mathbb{T}^2$. Then $\partial_1\partial_2 G$ and $\partial_1^2 G - \partial_2^2 G$ belong to L^∞. Equivalently,* (3) *holds.*

Proof. Let $\varphi \in C_c^\infty(B(0,1/2))$ with $\varphi = 1$ in $B(0,1/4)$. Then $H = \varphi F$ may be identified with a map on $\mathbb{T}^2$. Since $\Delta^2(G-H) \in C^\infty$, we have $G - H \in C^\infty$. We conclude via $\partial_1\partial_2 H$, $\partial_1^2 H - \partial_2^2 H \in L^\infty$.

Noting that, up to a constant, we have $G = \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(m_1^2+m_2^2)^2} e^{\imath m\cdot x}$, we find that Corollary 2 is equivalent to (3). □

Remark 1. Corollary 2 implies (6) when $n = 2$. Here is the proof. We treat, e.g., the first integral in (6). We have

$$\int \partial_1 u \partial_2 u = \int \left[\left(\Delta^2 G\right) * \partial_1 u\right]\partial_2 u = -\int \left[(\partial_1\partial_2 G) * (\Delta u)\right](\Delta u).$$

We deduce that

$$\left|\int \partial_1 u \partial_2 u\right| \leqslant \left\|(\partial_1\partial_2 G) * (\Delta u)\right\|_{L^\infty} \|\Delta u\|_{L^1} \leqslant \|\partial_1\partial_2 G\|_{L^\infty} \|\Delta u\|_{L^1} \|\Delta u\|_{L^1} \lesssim \|\Delta u\|_{L^1}^2.$$

Next we discuss the higher dimensional analogs of Proposition 1 and Corollary 2, as well as their connection to (5) and (6).

Proposition 3. *In $\mathbb{R}^n$, the operator $(-\Delta)^{n/2+1}$ has a fundamental solution F such that $\partial_1\partial_2 F \in L^\infty$ and $\partial_1^2 F - \partial_2^2 F \in L^\infty$.*

Here, when n is odd, a fundamental solution F is a temperate solution of $(-\Delta)^{n/2+1/2} F = \mathcal{F}^{-1}((2\pi|\xi|)^{-1})$.

Proof. One may check that, with $\alpha_n := \frac{1}{2^{n+1}\pi^{n/2}\Gamma(n/2+1)}$, the map $F(x) := \alpha_n\{|x|^2 \ln|x| - |x|^2/2\}$ is a fundamental solution. In addition, we have $\partial_1\partial_2 F = 2\alpha_n x_1 x_2 |x|^{-2}$ and $\partial_1^2 F - \partial_2^2 F = 2\alpha_n (x_1^2 - x_2^2)|x|^{-2}$. □

The analogs of Corollary 2 and formula (3) are given by

Proposition 4. *Let G be the (unique up to constants) solution of $(-\Delta)^{n/2+1} G = \delta - (1/2\pi)^n$ on $\mathbb{T}^n$. Then $\partial_1\partial_2 G$ and $\partial_1^2 G - \partial_2^2 G$ belong to L^∞. Equivalently,*

$$\sum_{m\in\mathbb{Z}^n\setminus\{0\}} \frac{m_1 m_2}{|m|^{n+2}} e^{\imath m\cdot x} \in L^\infty \quad \text{and} \quad \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \frac{m_1^2 - m_2^2}{|m|^{n+2}} e^{\imath m\cdot x} \in L^\infty. \tag{7}$$

Sketch of proof. When n is even, $(-\Delta)^{n/2+1}$ is a local operator, so that we may repeat the proof of Corollary 2. When n is odd, we mimic the proof of (3) in [1, pp. 405–406]. □

Remark 2. In the same way that Corollary 2 implies (6) when $n = 2$, Proposition 4 implies (6) for all n.

Remark 3. One can recover estimate (5) of Maz'ya by combining (6) to some arguments used by Bourgain and Brezis [1] in the proof of (2). The starting point of the proof is the following estimate, reminiscent of [1, p. 404], and valid when u has zero average:

$$\|u\|_{H^{1-n/2}}^2 \sim \sum_{j<k} \left(\left\|\partial_j\partial_k(-\Delta)^{-n/4-1/2}u\right\|_{L^2}^2 + \left\|\left(\partial_j^2 - \partial_k^2\right)(-\Delta)^{-n/4-1/2}u\right\|_{L^2}^2 \right). \tag{8}$$

Let $\nabla u = \boldsymbol{U} + \boldsymbol{V}$, $\boldsymbol{U} \in H^{-n/2}$, $\boldsymbol{V} \in L^1$. Inspired by [1, pp. 403–405], we treat, e.g., the first term in (8) for $j = 1$, $k = 2$ using the identity

$$\begin{aligned} \left\|\partial_1\partial_2(-\Delta)^{-n/4-1/2}u\right\|_{L^2}^2 = \int \Big\{ & \left[\partial_1\partial_2(-\Delta)^{-n/4-1/2}u\right]\left[(-\Delta)^{-n/4-1/2}(\partial_1 U_2 + \partial_2 U_1)\right] \\ & - \left[\partial_1(-\Delta)^{-n/4-1/2}U_2\right]\left[\partial_2(-\Delta)^{-n/4-1/2}U_1\right] \\ & + \left[\partial_1(-\Delta)^{-n/4-1/2}V_2\right]\left[\partial_2(-\Delta)^{-n/4-1/2}V_1\right]\Big\}. \end{aligned} \tag{9}$$

Using standard elliptic estimates for the first two integrals on the right-hand side of (9) and Proposition 4 for the last integral, we find that

$$\left\|\partial_1\partial_2(-\Delta)^{-n/4-1/2}u\right\|_{L^2}^2 \lesssim \left\|\partial_1\partial_2(-\Delta)^{-n/4-1/2}u\right\|_{L^2} \|U\|_{H^{-n/2}} + \|U\|_{H^{-n/2}}^2 + \|V\|_{L^1}^2,$$

i.e., $\|u\|_{H^{1-n/2}} \lesssim \|U\|_{H^{-n/2}} + \|V\|_{L^1}$.

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