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Probability Theory

On the integral representation of g-expectations

Sur le représentation intégrale pour les g-espérances

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ABSTRACT

In this Note, we give a necessary and sufficient condition on deterministic g under which g-expectations can be represented as Choquet expectations.

RÉSUMÉ

Dans cette Note, nous donnons une condition nécessaire et suffisante sur g déterministe sous laquelle les g-espérances peut être représentée par les espérances de Choquet.

1. Introduction

Peng [14] introduced the notions of g-expectations and conditional g-expectations via a class of backward stochastic differential equations (BSDEs), and showed that g-expectations are dynamically consistent nonlinear expectations. Since then, many researchers have been investigating the properties of g-expectations and their connection with other fields (see [1–5,7,9–14] and the references therein). In [2] and [5], Chen et al. studied an integral representation problem: if a g-expectation can be represented as a Choquet expectation, can we find the form of the generator g? For 1-dimensional Brownian motion case, they gave a necessary and sufficient condition on g. But their method cannot be used for multi-dimensional case. In this Note, we give a more simple and direct method to deal with this problem, especially for multi-dimensional case.

2. Preliminaries

Let $(W_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion defined on a completed probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\geqslant 0}$ be the natural filtration generated by this Brownian motion. Fix $T > 0$. We denote by $L^2(\mathcal{F}_t)$, $t \in [0, T]$, the set of all square integrable $\mathcal{F}_t$-measurable random variables and $L^2(0, T; \mathbb{R}^n)$ the space of all progressively measurable, $\mathbb{R}^n$-valued processes $(v_t)_{t\in[0,T]}$ with $E\int_0^T |v_t|^2\,\mathrm{d}t < \infty$.

In this Note, we consider a deterministic function $g : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $t \mapsto g(t, y, z)$ is measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. For the function g, we will use the following assumptions:

(H1) There exists a constant $K \geqslant 0$ such that

$$\left|g(t, y, z) - g\left(t, y', z'\right)\right| \leqslant K\left(\left|y - y'\right| + \left|z - z'\right|\right), \quad \forall t, y, y', z, z'.$$

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(H2) $g(t, y, 0) \equiv 0$ for all $(t, y) \in [0, T] \times \mathbb{R}$.
(H3) For each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $t \mapsto g(t, y, z)$ is continuous.

Under the assumptions (H1) and (H2), Pardoux and Peng [12] showed that for each $\xi \in L^2(\mathcal{F}_T)$, the BSDE

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}W_s, \quad t \in [0, T], \tag{1}$$

has a unique solution $(y_t, z_t)_{t\in[0,T]} \in L^2(0, T; \mathbb{R}^{1+d})$. Using the solution of BSDE (1), Peng [14] proposed the following notions:

Definition 2.1. Suppose g satisfies (H1) and (H2). For each $\xi \in L^2(\mathcal{F}_T)$, let $(y_t, z_t)_{t\in[0,T]}$ be the solution of BSDE (1), define

$$\mathcal{E}_g[\xi \mid \mathcal{F}_t] := y_t \quad \text{for } t \in [0, T].$$

$\mathcal{E}_g[\xi \mid \mathcal{F}_t]$ is called the conditional g-expectation of ξ with respect to $\mathcal{F}_t$. In particular, if $t = 0$, we write $\mathcal{E}_g[\xi]$ which is called the g-expectation of ξ.

Let g satisfy (H1) and (H2). The g-probability $P_g(\cdot)$ is defined by

$$P_g(A) := \mathcal{E}_g[I_A] \quad \text{for all } A \in \mathcal{F}_T.$$

The related Choquet expectation (see [6]) is denoted by $\mathcal{C}_g$, i.e.,

$$\mathcal{C}_g[\xi] = \int_{-\infty}^0 \big[P_g(\xi \geqslant t) - 1\big]\,\mathrm{d}t + \int_0^\infty P_g(\xi \geqslant t)\,\mathrm{d}t \quad \text{for } \xi \in L^2(\mathcal{F}_T).$$

Two random variables ξ and η are called comonotonic if

$$\big[\xi(\omega) - \xi(\omega')\big]\big[\eta(\omega) - \eta(\omega')\big] \geqslant 0 \quad \text{for each } \omega, \omega' \in \Omega.$$

The following properties of $\mathcal{C}_g$ can be found in the book [8].

(1) Monotonicity: If $\xi \geqslant \eta$, then $\mathcal{C}_g[\xi] \geqslant \mathcal{C}_g[\eta]$.
(2) Positive homogeneity: If $\lambda \geqslant 0$, then $\mathcal{C}_g[\lambda\xi] = \lambda\mathcal{C}_g[\xi]$.
(3) Translation invariance: If $c \in \mathbb{R}$, then $\mathcal{C}_g[\xi + c] = \mathcal{C}_g[\xi] + c$.
(4) Comonotonic additivity: If ξ and η are comonotonic, then $\mathcal{C}_g[\xi + \eta] = \mathcal{C}_g[\xi] + \mathcal{C}_g[\eta]$.

3. Main result

Now we give the main result:

Theorem 3.1. *Suppose g satisfies* (H1)–(H3)*. Then the g-expectation can be represented as the Choquet expectation if and only if the g-expectation is the classical linear expectation.*

Remark 1. For the case $d = 1$, the above theorem is the main result in Chen et al. [2].

For proving this theorem, we need the following lemma, which is a direct consequence of Theorem 4.7 in [13]. We always use the notation $W_t = (W_t^1, \ldots, W_t^d)$.

Lemma 3.2. *Suppose that $d = 2$ and g satisfies* (H1)–(H3)*. Then for each $a, b, \lambda \in \mathbb{R}$, we have*

$$\mathcal{E}_g[I_{[W_T^1 \geqslant a]} + \lambda I_{[W_T^2 \geqslant b]} \mid \mathcal{F}_t] = \mathcal{E}_g[I_{[W_T^1 - W_t^1 \geqslant a-x]} + \lambda I_{[W_T^2 - W_t^2 \geqslant b-y]}]|_{(x,y)=(W_t^1, W_t^2)}.$$

Sketch of proof of Theorem 3.1. The sufficient condition is obvious. We now prove the necessary condition. For simplicity, we only prove the case $d = 2$ (see [10] for general case). It follows from $\mathcal{E}_g = \mathcal{C}_g$ and the properties of translation invariance and positive homogeneity of $\mathcal{C}_g$ that

$$\mathcal{E}_g[\xi + c] = \mathcal{E}_g[\xi] + c \quad \text{for each } c \in \mathbb{R}; \qquad \mathcal{E}_g[\lambda\xi] = \lambda\mathcal{E}_g[\xi] \quad \text{for each } \lambda \geqslant 0.$$

Thus, by Theorems 3.1 and 3.4 in Jiang [11] (see also [1] and [2]), we obtain that g is independent of y and $g(t, \lambda z) = \lambda g(t, z)$ for each $\lambda \geqslant 0$. On the other hand, note that $(1 - \lambda)I_{[W_T^1 - W_t^1 \geqslant a]}$ and $\lambda(I_{[W_T^1 - W_t^1 \geqslant a]} + I_{[W_T^2 - W_t^2 \geqslant b]})$ are comonotonic for each $\lambda \in (0,1)$, $a, b \in \mathbb{R}$, then, by Lemma 3.2 and the comonotonic additivity of $\mathcal{C}_g$, we get the following key relation: for each $\lambda \in (0,1)$, $t \in [0,T]$, $n \in \mathbb{N}$,

$$\mathcal{E}_g[I_{[W_T^1 \geqslant n]} + \lambda I_{[W_T^2 \geqslant 0]} \mid \mathcal{F}_t] = \lambda \mathcal{E}_g[I_{[W_T^1 \geqslant n]} + I_{[W_T^2 \geqslant 0]} \mid \mathcal{F}_t] + (1-\lambda)\mathcal{E}_g[I_{[W_T^1 \geqslant n]} \mid \mathcal{F}_t]. \tag{2}$$

Let $(y_t^{\lambda,n}, z_t^{\lambda,n})_{t\in[0,T]}$, $(\tilde{y}_t^n, \tilde{z}_t^n)_{t\in[0,T]}$ and $(\hat{y}_t^n, \hat{z}_t^n)_{t\in[0,T]}$ be the solutions of BSDE (1) corresponding to the terminal values $I_{[W_T^1 \geqslant n]} + \lambda I_{[W_T^2 \geqslant 0]}$, $I_{[W_T^1 \geqslant n]} + I_{[W_T^2 \geqslant 0]}$ and $I_{[W_T^1 \geqslant n]}$, respectively. By (2), we have $y_t^{\lambda,n} = \lambda \tilde{y}_t^n + (1-\lambda)\hat{y}_t^n$. From this and g is independent of y, we deduce that for each $\lambda \in (0,1)$,

$$\mathrm{d}P \times \mathrm{d}t - \text{a.s.}, \quad g\big(t, \lambda \tilde{z}_t^n + (1-\lambda)\hat{z}_t^n\big) = \lambda g\big(t, \tilde{z}_t^n\big) + (1-\lambda) g\big(t, \hat{z}_t^n\big).$$

Since $\lambda \in (0,1)$ is arbitrary and g is positively homogeneous in z, we conclude that for each $l \geqslant 0$,

$$\mathrm{d}P \times \mathrm{d}t - \text{a.s.}, \quad g\big(t, \tilde{z}_t^n + l\hat{z}_t^n\big) = g\big(t, \tilde{z}_t^n\big) + g\big(t, l\hat{z}_t^n\big). \tag{3}$$

It follows from Theorem 1 in [2] that $g(t, z_1, 0) = g(t, 1, 0) z_1$ for each $z_1 \in \mathbb{R}$, $t \in [0,T]$. Then we have

$$\mathrm{d}P \times \mathrm{d}t - \text{a.s.}, \quad \hat{z}_t^n = \left(\frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(n - W_t^1 - \int_t^T g(s,1,0)\,\mathrm{d}s)^2}{2(T-t)}\right), 0\right). \tag{4}$$

Combining (3) with (4), we obtain that for each $p \geqslant 0$,

$$\mathrm{d}P \times \mathrm{d}t - \text{a.s.}, \quad g\big(t, \tilde{z}_t^n + (p,0)\big) = g\big(t, \tilde{z}_t^n\big) + g(t, p, 0). \tag{5}$$

Let $(\bar{y}_t, \bar{z}_t)_{t\in[0,T]}$ be the solution of BSDE (1) corresponding to the terminal value $I_{[W_T^2 \geqslant 0]}$. Noting that $I_{[W_T^1 \geqslant n]} + I_{[W_T^2 \geqslant 0]} \to I_{[W_T^2 \geqslant 0]}$ in $L^2(\mathcal{F}_T)$, by Theorem 2.3 in [13] (see also [1] and [9]), we have $(\tilde{z}_t^n)_{t\in[0,T]} \to (\bar{z}_t)_{t\in[0,T]}$ in $L^2(0,T;\mathbb{R}^2)$. Since g satisfies Lipschitz assumption (H1), we get for each $p \geqslant 0$,

$$g\big(t, \tilde{z}_t^n + (p,0)\big) \to g\big(t, \bar{z}_t + (p,0)\big) \quad \text{in } L^2(0,T;\mathbb{R}).$$

This with (5) implies that for each $p \geqslant 0$,

$$\mathrm{d}P \times \mathrm{d}t - \text{a.s.}, \quad g\big(t, \bar{z}_t + (p,0)\big) = g(t, \bar{z}_t) + g(t, p, 0). \tag{6}$$

Also, by $g(t, 0, z_2) = g(t, 0, 1) z_2$, we have

$$\mathrm{d}P \times \mathrm{d}t - \text{a.s.}, \quad \bar{z}_t = \left(0, \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(W_t^2 + \int_t^T g(s,0,1)\,\mathrm{d}s)^2}{2(T-t)}\right)\right). \tag{7}$$

Thus, by (6), (7) and $g(t, \lambda z) = \lambda g(t, z)$ for each $\lambda \geqslant 0$, we obtain that for each $z_1 \geqslant 0$, $z_2 \geqslant 0$,

$$g(t, z_1, z_2) = g(t, 1, 0) z_1 + g(t, 0, 1) z_2. \tag{8}$$

Similarly, we can obtain (8) for each $(z_1, z_2) \in \mathbb{R}^2$. Then by the Girsanov Theorem, $\mathcal{E}_g$ is the classical linear expectation. The proof is completed. □

Remark 2. The key relation (2) is not trivial. Because the comonotonic additivity of g-expectation does not imply the comonotonic additivity of conditional g-expectation, which is discussed in detail in [2]. Moreover, our method also holds without the continuous assumption (H3) on g (see [10]).

Remark 3. In [7], the authors showed that a class of dynamically consistent nonlinear expectations must be g-expectations. So, our result also indicates that Choquet expectations cannot be dynamically consistent in the sense in [7].

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