Algebra/Group Theory

Pointed Hopf algebras over some sporadic simple groups

Algèbres de Hopf pointées sur quelques groupes simples sporadiques

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1. Introduction

Let $k$ be an algebraically closed field of characteristic 0. In this Note, we announce a new contribution to the classification of finite-dimensional Hopf algebras over $k$. As is known, different classes of finite-dimensional Hopf algebras have to be studied separately because the pertaining methods are radically different. There is a method for pointed Hopf algebras (those whose coradical is a group algebra $kG$) that has been applied with satisfactory results when $G$ is Abelian [8]; an exposition of the method can be found in [7]. Recently, it appeared that many finite simple (or almost simple) groups $G$ admit very few finite-dimensional, pointed Hopf algebras with coradical isomorphic to $kG$:

- Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to $A_m$, $m \geq 5$, is a group algebra [2].
- Same for the groups $SL(2, 2^n)$, $n > 1$ [10] and $M_{20}, M_{21} = PSL(3, 4)$ [11].
- Most of the pointed Hopf algebras over the symmetric groups have infinite dimension, with the exception of a short list of open possibilities, see [2,4] and references therein. More precisely, most of the irreducible Yetter–Drinfeld modules have infinite-dimensional Nichols algebras (see below).

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This is a report on finite-dimensional pointed Hopf algebras over sporadic simple groups. As part of our results, we have the following:

**Theorem 1.** Let \( G \) be any sporadic simple group, different from the Fischer group \( Fi_{22} \), the Baby Monster \( B \) and the Monster \( M \). If \( H \) is a finite-dimensional pointed Hopf algebra with \( G(H) \cong G \), then \( H \cong kG \).

The Theorem holds more generally over any field of characteristic 0, since the property of being pointed is stable under extension of scalars.

### 1.1. Glossary

For the reader’s convenience, we recall a few definitions that are central to our work. More information can be found in [5,7]. Let \( H \) be a Hopf algebra with comultiplication \( \Delta \) and bijective antipode \( S \).

- An element \( g \neq 0 \) in \( H \) is a *grouplike* if \( \Delta(g) = g \otimes g \); the set of all grouplikes is a group \( G(H) \) with multiplication given by the product of \( H \).
- A Yetter–Drinfeld module over \( H \) is a left \( H \)-module \( M \) that bears also a structure \( \lambda : M \rightarrow H \otimes M \) of \( H \)-comodule, compatible with the action in an appropriate sense. If \( H \) is finite-dimensional, then a Yetter–Drinfeld module is the same as a module over the Drinfeld double of \( H \). For instance, if \( H = kG \) is the group algebra of a finite group \( G \), then a Yetter–Drinfeld module over \( H \) is a left \( G \)-module \( M \) that bears also a \( G \)-gradation \( M = \bigoplus_{g \in G} M_g \), compatibility meaning that \( h : M_g = M_{gh^{-1}} \) for all \( g, h \in G \).
- A rack is a pair \((X, \cdot)\) where \( X \) is a non-empty set and \( \cdot : X \times X \rightarrow X \) is an operation such that the map \( \varphi_x = x \cdot \_ \) is bijective for any \( x \in X \), and \( x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) \) for all \( x, y, z \in X \). A map \( q : X \times X \rightarrow GL(n, k) \) is a 2-cocycle of degree \( n \) if
  \[
  q_{x, y, z} q_{y, z} = q_{x, y, z/q} q_{x, y, z} \quad \text{for all } x, y, z \in X.
  \]
- A braided vector space is a pair \((V, c)\) where \( V \) is a vector space and \( c \in GL(V \otimes V) \) fulfills the braid equation: \((c \otimes id)(id \circ c)(c \otimes id) = (id \circ c)(c \otimes id)(id \circ c)\). Examples:
  (i) Any Yetter–Drinfeld module is a braided vector space in a natural way.
  (ii) Let \( X \) be a finite rack, \( q \) a 2-cocycle of degree \( n \), \( V = \mathbb{k}X \otimes \mathbb{k}^n \), where \( \mathbb{k}X \) is the vector space with basis \( e_x \), \( x \in X \).

We denote \( e_x v := e_x \otimes v \). Consider the linear isomorphism \( c^n : V \otimes V \rightarrow V \otimes V \), \( c^n(e_x \otimes e_y w) = e_x \otimes c^n(q_{x, y, z}) (w) \otimes e_y \).
- The braided vector spaces arising as Yetter–Drinfeld modules over group algebras of finite groups can be presented in terms of racks and cocycles, see a bit more of information below.
- We assume the reader familiar with the important notion of the *Nichols algebra* of a braided vector space, discussed at length in [7]. In short, one of the possible definitions of the Nichols algebra \( \mathfrak{B}(V) \) of a braided vector space \((V, c)\) is as follows. Since \( c \) satisfies the braid equation, it induces a representation of the braid algebra \( \mathfrak{B}_n \), \( \rho_n : \mathfrak{B}_n \rightarrow GL(V \otimes \mathbb{k}^n) \), for each \( n \geq 2 \). Let \( \mathfrak{B}_n = \sum_{\sigma \in S_n} \rho_n(M(\sigma)) \in End(V \otimes \mathbb{k}^n) \), where \( M : \mathfrak{S}_n \rightarrow \mathfrak{B}_n \) is the so-called Matsumoto section (not a morphism of groups, but preserves product when length is preserved). Then \( \mathfrak{B}(V) \) is the quotient of the tensor algebra \( T(V) \) by \( \bigoplus_{n \geq 2} \ker \rho_n \), in fact a 2-sided ideal of \( T(V) \). If \( c \) is the usual switch, then \( \mathfrak{B}(V) \) is just the symmetric algebra of \( V \); but in general the determination of a Nichols algebra is quite a difficult task.

### 2. Outline of the proof

A complete proof of Theorem 1 for the groups \( M_{22} \) and \( M_{24} \) is contained in [9]; the proof for the other groups is included in [3].

We sketch now the proof in two main reductions. The first one has been explained in several places, with detail in [7], but we include a brief summary for completeness. We remind that if \( U \) is a braided vector subspace of \( V \), then \( \mathfrak{B}(U) \hookrightarrow \mathfrak{B}(V) \).

#### 2.1. A general reduction

Let \( G \) be a finite group, \( H \) a pointed Hopf algebra with \( G(H) \cong G \). Then there are two basic invariants of \( H \), a Yetter–Drinfeld module \( V \) over \( kG \) (called the infinitesimal braiding of \( H \)) and its Nichols algebra \( \mathfrak{B}(V) \). We have \( |G| \dim \mathfrak{B}(V) \leq \dim H \). Therefore, the following statements are equivalent:

1. If \( H \) is a finite-dimensional pointed Hopf algebra with \( G(H) \cong G \), then \( H \cong kG \).
2. If \( V \neq 0 \) is a Yetter–Drinfeld module over \( kG \), then \( \dim \mathfrak{B}(V) = \infty \).
3. If \( V \) is an irreducible Yetter–Drinfeld module over \( kG \), then \( \dim \mathfrak{B}(V) = \infty \).
2.2. Looking at subracks

We focus on (3) above. The second reduction has been the basis of our recent papers. It starts from the well-known classification of irreducible Yetter–Drinfeld modules over \( kG \) by pairs \((\mathcal{O}, \rho)\), where \( \mathcal{O} \) is a conjugacy class in \( G \) and \( \rho \) is an irreducible representation of the stabilizer \( G^s \) of a fixed point \( s \in \mathcal{O} \). Now, the definition of the Nichols algebra \( \mathfrak{B}(\mathcal{O}, \rho) \) of the corresponding Yetter–Drinfeld module \( M(\mathcal{O}, \rho) \) just depends on the braiding. If \( \dim \rho = 1 \), then this braiding depends only on the rack \( \mathcal{O} \) and a 2-cocycle \( q : \mathcal{O} \times \mathcal{O} \to k^\times \) [5]. Namely, \( \mathcal{O} \) is a rack with the product \( xy := yx^{-1} \). \( M(\mathcal{O}, \rho) \) has a natural basis \( (e_x)_{x \in \mathcal{O}} \) and the braiding is given by \( c(e_x \otimes e_y) = q_{xy} e_{yx} \otimes e_x \). If there exists a subrack \( X \) of \( \mathcal{O} \) such that the Nichols algebra of the braided vector space defined by \( X \) and the restriction of \( q \) is infinite dimensional, then \( \dim \mathfrak{B}(\mathcal{O}, \rho) = \infty \).

We recall some examples of racks which are relevant in this work.

(i) Abelian racks: those racks \( X \) such that \( x \mapsto y = y \) for all \( x, y \in X \).
(ii) \( D_p \): the class of involutions in the dihedral group \( \mathbb{D}_p \) (of order 2p), \( p \) a prime.
(iii) \( O \): the class of 4-cycles in \( S_4 \).
(iv) Doubles of racks: if \( X \) is a rack, \( \rho \) a decomposable subrack, then \( X^{(2)} \) denotes the disjoint union of two copies of \( X \) each acting on the other by left multiplication.

We are interested in finding subracks which are Abelian, or isomorphic to \( D_p^{(2)} \) or to \( O^{(2)} \), by the following reasons:

(A) If \( X \) is Abelian, then the corresponding braided vector space is of diagonal type. Braided vector spaces of diagonal type with finite-dimensional Nichols algebra where classified in [13]; thus, we just need to check if the matrix \( (q_{xy}) \) belongs or not to the list in [13].

(B) If \( X \) is isomorphic either to \( D_p^{(2)} \) or to \( O^{(2)} \), then for some specific cocycles, the related Nichols algebras have infinite dimension [6, Ths. 4.7, 4.8].

Variations:

(a) If \( \dim \rho > 1 \), similar arguments apply.
(b) Sometimes the rack \( X \) is not Abelian, but the braided vector space produced by \( X \) and the 2-cocycle can be realized with an Abelian rack, by a suitable change of basis.
(c) Let \( F < G \) be a subgroup, \( s \in F, \mathcal{O}^F \) resp. \( \mathcal{O}^G \) the conjugacy class of \( s \) in \( F \), resp. in \( G \). If \( \dim \mathfrak{B}(\mathcal{O}^F, \tau) = \infty \) for any irreducible representation \( \tau \) of \( F \), then \( \dim \mathfrak{B}(\mathcal{O}^G, \rho) = \infty \) for any irreducible representation \( \rho \) of \( G \).
(d) A conjugacy class \( \mathcal{O} \) is real if \( \mathcal{O} = \mathcal{O}^{-1} \). It is quasireal if \( \mathcal{O} = \mathcal{O}^m \) for some integer \( m \), \( 1 < m < N \), where \( N \) is the order of the elements in \( \mathcal{O} \). The search of subracks isomorphic to \( D_p^{(2)} \) or to \( O^{(2)} \), as well as the verification that the restriction of the cocycle \( q \) is as needed in (2.2), is greatly simplified in a real (quasireal) conjugacy class [1].
(e) We say that a rack \( X \) is of type \( D \) if there exists a decomposable subrack \( Y = R [ \bigcup \mathcal{S} ] \) of \( X \) such that \( r \cdot (s \cdot (r \cdot s)) \neq s \), for some \( r \in R, s \in \mathcal{S} \). If a conjugacy class \( \mathcal{O} \) is a rack of type \( D \), then \( \dim \mathfrak{B}(\mathcal{O}, \rho) = \infty \) for any \( \rho \) (see [2] and Theorem 8.6 of [14]).

2.3. Computations

We now fix a sporadic group \( G \) as in Theorem 1. We extracted relevant information from the ATLAS [15] with the AtlasRep package [16]. Then, we checked when a conjugacy class is real or quasireal or of type \( D \). We used GAP [12] for the computations.

These tools allow us to apply the techniques sketched above to all pairs \((\mathcal{O}, \rho)\) and establish the validity of (2.1).

2.4.

Some of these results were announced in several meetings:

- First De Brün Workshop on Computational Algebra, National University of Ireland, Galway, Ireland. August, 2008
- Groupes quantiques dynamiques et catégories de fusion. CIRM, Luminy, France. April 2008.

References