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Group Theory

Spectral gaps in $SU(d)$

Trou spectral dans $SU(d)$

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ABSTRACT

It is shown that if $g_1, \ldots, g_k$ are algebraic elements in $SU(d)$ generating a dense subgroup, then the corresponding Hecke operator has a spectral gap.

RÉSUMÉ

On démontre que si $g_1, \ldots, g_k$ sont des éléments algébriques de $SU(d)$ et le groupe engendré par $g_1, \ldots, g_k$ est dense, alors l'opérateur de Hecke défini par ces éléments a un trou spectral.

Version française abrégée

Soit $g_1, \ldots, g_k \in SU(d) \cap \mathrm{Mat}_{d\times d}(\bar{\mathbb{Q}})$ et $\Gamma = \langle g_1, \ldots, g_k \rangle$ le groupe engendré par $g_1, \ldots, g_k$. Supposons Γ dense dans $SU(d)$.

Théorème. *L'opérateur de Hecke*

$$Tf(x) = \frac{1}{2k} \sum_{1 \leqslant j \leqslant k} \left(f(g_j x) + f\left(g_j^{-1} x\right)\right)$$

a un trou spectral.

Ceci généralize le résultat antérieur [4] pour $SU(2)$. L'approche suivie ici diffère cependant et elle est essentiellement analogue à celle de [5] pour les groupes $SL_d(p^n)$ avec p fixé et $n \to \infty$. Des techniques d'arithmétique combinatoire, de la theorie des représentations et produits aléatoires de matrices y sont utilisées.

1. We assume $g_1, \ldots, g_k \in SU(d) \cap \mathrm{Mat}_{d\times d}(\bar{\mathbb{Q}})$ and denote $\Gamma = \langle g_1, \ldots, g_k \rangle$ the generated group. Assume further that Γ is Zariski dense in SL_d or, equivalently, that Γ is topologically dense in $SU(d)$.

Denote

$$(Tf)(x) = \frac{1}{2k} \sum_{j=1}^{k} \left(f(g_j x) + f\left(g_j^{-1} x\right)\right)$$

the corresponding Hecke operator acting on $L^2(G)$, $G = SU(d)$.

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Theorem 1. *T has a spectral gap.*

The result for $d=2$ was obtained in [4]. As in [4], we rely on methods from arithmetic combinatorics. But the approach followed here is significantly different from that of [4] and resembles that of [5] on expansion in groups $SL_d(p^n)$ with p fixed and $n\to\infty$. Similarly to [5], the assumption of Zariski density is exploited through the theory of random matrix products (cf. [1]).

2. By a result of [6], we may take $k=2$ and assume $\{g_1,g_2\}$ free generators of the free group F_2. Define

$$\nu=\frac{1}{4}(\delta_{g_1}+\delta_{g_2}+\delta_{g_1^{-1}}+\delta_{g_2^{-1}})$$

the symmetric probability measure on G and denote $\nu^{(\ell)}$ its ℓ-fold convolution power. Set for $\delta>0$

$$P_\delta=\frac{1_{B(1,\delta)}}{|B(1,\delta)|}$$

providing an approximate identity on G.

Proposition 1. *There is $\kappa>0$ such that if G_1 is a non-trivial closed subgroup of G, then*

$$\nu^{(\ell)}(G_1)<e^{-\kappa\ell}\quad \text{for } \ell\to\infty. \tag{1}$$

The proof of this 'escape property' relies on our assumption that Γ is Zariski dense and results on random matrix products, that are applied in suitable exterior powers of the adjoint representation of G. As in [4], we establish the following 'flattening property':

Proposition 2. *Given $\tau>0$, there is a positive integer $\ell<C(\tau)\log\frac{1}{\delta}$ such that*

$$\left\|\nu^{(\ell)}*P_\delta\right\|_\infty<\delta^{-\tau}. \tag{2}$$

It is derived by straightforward iteration of

Lemma 1. *Given $\gamma>0$, there is $\kappa>0$ such that for $\delta>0$ small enough, $\ell\sim\log\frac{1}{\delta}$, if*

$$\left\|\nu^{(\ell)}*P_\delta\right\|_2>\delta^{-\gamma}. \tag{3}$$

Then

$$\left\|\nu^{(2\ell)}*P_\delta\right\|_2<\delta^{\kappa}\left\|\nu^{(\ell)}*P_\delta\right\|_\delta. \tag{4}$$

With Proposition 2 at hand, the proof of a spectral gap may then be completed by considerations from representation theory (the Sarnak–Xue argument, also used in [4], or variants).

3. Returning to Lemma 1, the first step is to apply T. Tao's version of the Balog–Szemeredi–Gowers lemma (cf. [7]) for compact groups. Denoting $\mu=\nu^{(\ell)}*P_\delta$ and assuming (4) fails, one obtains a subset $H\subset G$, H a union of δ-balls, and a finite subset X of G such that

(5) $H=H^{-1}$,
(6) $H.H\subset H.X\cap X.H$,
(7) $|X|<\delta^{-\varepsilon}$,
(8) $\mu(aH)>\delta^{\varepsilon}$ for some $a\in G$,
(9) $|H|<\delta^{\gamma}$

(here $\varepsilon>0$ is an arbitrary small, fixed number and $|\ |$ is used in (7) to denote 'cardinality' and in (9) for 'Haar-measure').

Recall that (5)–(6) mean that H is an 'approximate group' (cf. [7]). The goal is to show that properties (5)–(9) are not compatible and get a contradiction.

4. Next we specify some technical ingredients.

Crucial use is made of the 'discretized ring theorem' (see [2,3]). The version needed here is the following

Proposition 3. *Given $\sigma > 0$, there is $\gamma > 0$ such that if $\delta > 0$ is small enough and $A \subset \mathbb{C}^d \cap B(0,1)$ satisfies*

$$N(A, \delta) > \delta^{-\sigma} \tag{10}$$

then there is $\xi \in \mathbb{C}^d$, $|\xi| = 1$ such that

$$\big[0, \delta^\gamma\big]\xi \subset A' + B\big(0, \delta^{\gamma+1}\big). \tag{11}$$

Here A' denotes a 'sum-product' set $s_1 A^{(s_2)} - s_1 A^{(s_2)}$ of A, with s_1, s_2 bounded in terms of σ.

In (10), $N(A, \delta)$ refers to the metrical entropy, i.e. the minimum number of δ-balls needed to cover A. We used the notations $s\mathcal{A} = \underbrace{\mathcal{A} + \cdots + \mathcal{A}}_{s\text{-fold}}$ and $\mathcal{A}^{(s)} = \underbrace{\mathcal{A} \cdots \mathcal{A}}_{s\text{-fold}}$ for the s-fold sum (resp. product) sets.

Proposition 3 is derived from the following result that generalizes [3]:

Theorem 2. *Let $A \subset [0,1]^d$ satisfy*

$$N(A, \delta) = \delta^{-\sigma} \quad (0 < \sigma < d) \tag{12}$$

and also a non-concentration property

$$N(A \cap I, \delta) < c\delta_1^\kappa N(A, \delta) \quad \text{if } \delta < \delta_1 < 1 \text{ and } I \text{ any } \delta_1\text{-ball}. \tag{13}$$

Let μ be a probability measure on $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$\|b\| \leqslant 1 \quad \text{for } b \in \operatorname{supp} \mu,$$
$$\max_{|v|=1=|w|} \mu\big[\big|\langle bv, w\rangle\big| < \delta_1\big] < \delta_1^\kappa \quad \text{if } \delta < \delta_1 < 1. \tag{14}$$

Then, for some $b \in \operatorname{supp} \mu$

$$N(A + A, \delta) + N(A + bA, \delta) > \delta^{-\sigma-\tau} \tag{15}$$

with $\tau = \tau(\sigma, \kappa) > 0$.

In order to apply Proposition 3, we construct 'almost' diagonal sets of matrices, using the following:

Lemma 2. *Assume $\{g_1, g_2\}$ in $U(d) \cap \mathrm{Mat}_{d\times d}(\bar{\mathbb{Q}})$ generate a free group and let $H \subset W_\ell(g_1, g_2)$ ($=$ the set of 'words' or length $\leqslant \ell$) satisfy*

$$\log|H| > c\ell. \tag{16}$$

Then there is a subset A of a product set $H^{(s)}$, $s < C$ and $\delta > 0$ such that

(17) $\log \frac{1}{\delta} \sim \ell$.
(18) *The elements of A are δ-separated.*
(19) *In an appropriate orthonormal basis, we have*

$$\operatorname{dist}(x, \Delta) < \delta \quad \text{for } x \in A$$

where Δ denotes the set of diagonal matrices.

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