C. R. Acad. Sci. Paris, Ser. I 348 (2010) 713–716

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Combinatorics

A solution to one of Knuth's permutation problems

Une solution d'un problème de permutation de Knuth

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A R T I C L E   I N F O

Article history:
Received 18 May 2010
Accepted 28 May 2010
Available online 17 June 2010

Presented by Christophe Soulé

A B S T R A C T

We answer a problem posed recently by Knuth: an n-dimensional box, with edges lying on the positive coordinate axes and generic edge lengths $W_1 < W_2 < \cdots < W_n$, is dissected into $n!$ pieces along the planes $x_i = x_j$. We describe which pieces have the same volume, and show that there are C_n distinct volumes, where C_n denotes the nth Catalan number.

R É S U M É

Nous répondons à un problème posé récemment par Knuth dans le contexte suivant : une boîte de dimension n, dont les arêtes s'alignent en partant de l'origine sur les axes de coordonnées positives et sont de longueur générique $W_1 < W_2 < \cdots < W_n$, est découpée en $n!$ morceaux par les hyperplans $x_i = x_j$. Nous décrivons alors les morceaux qui ont même volume et nous montrons qu'il y a C_n volumes distincts où C_n désigne le n-ième nombre de Catalan.

1. Introduction

In a recent talk [4], D. Knuth posed the following problem. Consider the n-dimensional box $B = [0, W_1] \times \cdots \times [0, W_n]$, where $W_1 < W_2 < \cdots < W_n$. If π is a permutation in S_n, the symmetric group on n letters, define the region

$$C_\pi = \{x \in B \mid x_{\pi(1)} \geqslant x_{\pi(2)} \geqslant \cdots \geqslant x_{\pi(n)}\}.$$

In other words, we dissect B by cutting it along the planes $x_i = x_j$, for $1 \leqslant i < j \leqslant n$. Each C_π is a piece of this dissection. Let us view the volume of C_π as a polynomial in the W_i. How many distinct volumes are there amongst the C_π, and which C_π have the same volume?

See Fig. 1 for the case $n = 3$, in which C_{132} and C_{123} have the same volume and all others have distinct volumes. The left-hand image shows the original problem; the other two images show B being dissected further along the planes $x_i = W_j$, so that the volumes may be more easily computed.

Definition 1.1. Let $\mathcal{P}$ denote the set of all partitions. Let π be a permutation with matrix $[a_{ij}]$ acting on the right. Define $\psi : S_n \to \mathcal{P}$ to be the map which sends π to the partition whose Young diagram is

$$\{(i', j'):\ a_{ij} = 0 \text{ for all } i \leqslant i' \text{ and } j \leqslant j'\}.$$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.05.005

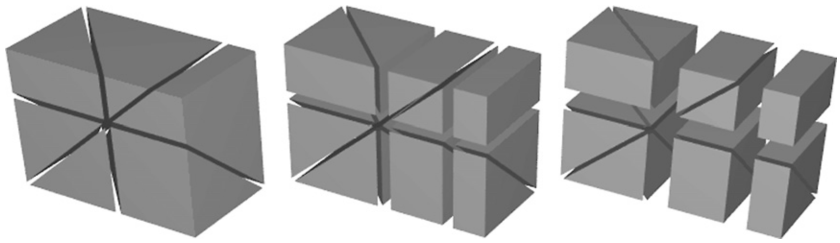

Fig. 1. Knuth's problem in dimension 3.

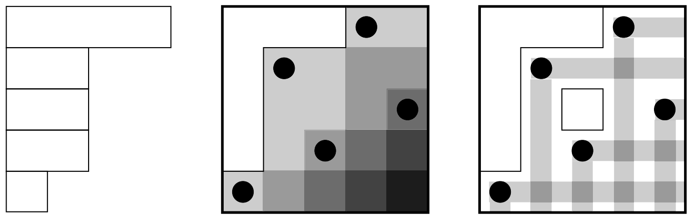

Fig. 2. Three of the constructions described in this article, applied to the permutation $\pi = 42\,531$. From left to right: $\lambda^{\max}(\pi) = (4,2,2,2,1)$, $\psi(\pi) = (3,1,1,1)$, and the diagram of π.

In other words, we cross out all matrix entries which lie weakly below and/or to the right of every one in the permutation matrix for π (see Fig. 2, center image). The entries which are not crossed out form the Young diagram of $\psi(\pi)$. Note that our permutation matrices always act on the right.

Theorem 1.2. *If π and σ are permutations, then* $\mathrm{Vol}(C_\pi) = \mathrm{Vol}(C_\sigma)$ *if and only if* $\psi(\pi) = \psi(\sigma)$.

We defer the proof of this theorem to the end of the paper. However, there is an immediate corollary, if we appeal to a few results in the literature:

Corollary 1.3. *The number of distinct elements of the set* $\{\mathrm{Vol}(C_\pi)\colon \pi \in S_n\}$ *is* $C_n = \frac{1}{n+1}\binom{2n}{n}$*, the nth Catalan number.*

Proof. Observe that $\psi(\pi)$ is closely related to a well-known construction, namely that of the *diagram* of the permutation π. To construct the diagram of π, one crosses out all entries *directly* below and *directly* to the right of each of the ones in the matrix for π. The result need not be a Young diagram (see Fig. 2, right image). As observed by Reifegerste [5], this procedure yields a Young diagram (and hence coincides with our $\psi(\pi)$) precisely when π is 132-avoiding. In other words, our ψ map yields precisely the rank-zero piece of Fulton's essential set [3,1]; the entire essential set has rank zero precisely when π is 132-avoiding. Alternatively, one can see directly that boundary of the Young diagram for $\psi(\pi)$ is always a Dyck path [2]. Both 132-avoiding permutations and Dyck paths are enumerated by the Catalan numbers. □

We do not know of a good reason why this problem, or our solution, should have anything to do with combinatorial representation theory; the map ψ as defined above arises naturally in our solution.

We note that Knuth's original setting of the problem [4] is slightly different. Namely, fix weights $W_1 < \cdots < W_n$, and let $X_1, \ldots, X_n$ be uniform random variables on $[0,1]$. We rank the quantities $x_i = W_i X_i$ from smallest to largest. If π is a permutation on n letters, define the event $E_\pi\colon x_{\pi(1)} \geqslant x_{\pi(2)} \geqslant \cdots \geqslant x_{\pi(n)}$. Knuth observed that when $n \geqslant 3$, certain of these events E_π occur with the same probability regardless of the choice of W_i. Theorem 1.2 now classifies the events $E(\pi)$ which occur with the same probability.

We would like to thank D. Knuth for helpful correspondence.

2. A refinement of the dissection

We will proceed in the manner suggested in Fig. 1: we subdivide the box B further, along the hyperplanes $x_i = W_j$. Once this is done, all pieces have very simple shapes, and are easily understood.

Definition 2.1. Let $W_0 = 0$, and define

$$a_i = W_i - W_{i-1} > 0 \quad \text{for } 1 \leqslant i \leqslant n,$$

$$\mathcal{B} = \{1,2,\ldots,n\}^n,$$

$$I = \{1\} \times \{1,2\} \times \{1,2,3\} \times \cdots \times \{1,2,3,\ldots,n\} \subseteq \mathcal{B}$$

and for $\rho = (\rho_1, \dots, \rho_n) \in \mathcal{B}$, we define the open box

$$B_\rho = (W_{\rho_1-1}, W_{\rho_1}) \times (W_{\rho_2-1}, W_{\rho_2}) \times \cdots \times (W_{\rho_n-1}, W_{\rho_n}).$$

Note that the dimensions of B_ρ are $a_{\rho_1} \times a_{\rho_2} \times \cdots \times a_{\rho_n}$. Observe that those boxes B_ρ for which $\rho \in I$ lie within B, and indeed partition B up to a set of volume zero (namely, the boundaries of the boxes). Also, note that if $\rho \in \mathcal{B}$ and $\rho_i = \rho_j$ for some $i \neq j$, then B_ρ is symmetric about the hyperplane $\{x_i = x_j\}$, whereas if $\rho_i < \rho_j$, then the hyperplane $\{x_i = x_j\}$ does not intersect B_ρ at all.

The motivation for all of these definitions is to simplify the computation of the volumes of the C_π. We begin with the following immediate observation:

Lemma 2.2. *For any $\rho \in \mathcal{B}$ and any $\pi \in S_n$, $\rho_{\pi(1)} \geqslant \rho_{\pi(2)} \geqslant \cdots \geqslant \rho_{\pi(n)}$ if and only if all points $x \in B_\rho$ satisfy $x_{\pi(1)} \geqslant x_{\pi(2)} \geqslant \cdots \geqslant x_{\pi(n)}$.*

The symmetric group S_n acts on $\mathcal{B}$ by permuting coordinates. Each box B_ρ has a stabilizer $G_\rho \leqslant S_n$ under this action. In fact, G_ρ is isomorphic to a product of symmetric groups

$$G_\rho \simeq S_{n_1} \times \cdots \times S_{n_k}$$

where n_j is the number of occurrences of the number j in ρ. Observe that G_ρ also acts faithfully on B_ρ by permuting coordinates, and so partitions B_ρ into $|G_\rho|$ equal-volume fundamental domains. We thus have the following volume computation:

Lemma 2.3.

$$\operatorname{Vol}(C_\pi \cap B_\rho) = \begin{cases} 0 & \text{if } C_\pi \cap B_\rho = \emptyset, \\ \frac{1}{|G_\rho|} a_{\rho_1} a_{\rho_2} \cdots a_{\rho_n} & \text{otherwise.} \end{cases}$$

3. Proof of the main theorem

For the following lemmata and their proofs, we adopt the following notation: Let $\rho = (\rho_1, \dots, \rho_n) \in \mathcal{B}$, $\pi \in S_n$, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{B}$ be such that $\lambda_i = \rho_{\pi(i)}$.

Lemma 3.1. *C_π meets B_ρ if and only if λ is a partition and $\lambda(i) \geqslant \pi(i)$.*

Proof. By Lemma 2.2, C_π meets B_ρ if and only if $\rho \in I$ and $\rho_{\pi(1)} \geqslant \cdots \geqslant \rho_{\pi(n)}$. Now, $\rho \in I \Leftrightarrow \rho_i \leqslant i \Leftrightarrow \lambda_i \leqslant \pi(i)$; similarly, $\rho_{\pi(1)} \geqslant \cdots \geqslant \rho_{\pi(n)}$ is equivalent to $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. □

Recall that the set of integer partitions forms a distributive lattice, *Young's lattice*, under the partial order of inclusion of Young diagrams. See, for example, [6, Section 7.2] for an introduction to Young's lattice.

Definition 3.2. Let $\lambda^{\max}(\pi) = \bigcup\{\mu \in \mathcal{P}\colon\ \mu \text{ is a partition with } n \text{ parts and } \mu_i \leqslant \pi(i)\}$, where $\bigcup$ denotes union of Young diagrams (the least upper bound in Young's lattice).

Lemma 3.3. *C_π meets B_ρ if and only if λ is a partition and $\lambda \subseteq \lambda^{\max}$ as Young diagrams.*

Proof. It is easy to check that if λ and μ are partitions which meet the condition of Lemma 3.1, then so is $\lambda \cup \mu$ (their union as Young diagrams). Moreover, if $\nu \subseteq \lambda$, then ν meets the conditions of Lemma 3.1. As such, the condition of Lemma 3.1 is equivalent to $\lambda \subseteq \lambda^{\max}$. □

Proof of Theorem 1.2. If λ is a partition, write $\rho(\lambda) = (\lambda_{\pi^{-1}(1)}, \dots, \lambda_{\pi^{-1}(n)})$. Taking $\rho = \rho(\lambda)$ and applying Lemmas 2.3 and 3.3, we see that

$$\operatorname{Vol}(C_\pi) = \sum_{\lambda \subseteq \lambda^{\max}(\pi)} \frac{1}{|G_{\rho(\lambda)}|} \prod_i a_{\lambda_i} = \sum_{\lambda \subseteq \lambda^{\max}(\pi)} \frac{1}{|G_\lambda|} \prod_i a_{\lambda_i}.$$

The latter equality holds because G_ρ is isomorphic to $G_{\sigma\cdot\rho}$ for any permutation $\sigma \in S_n$. As such, $\operatorname{Vol}(C_\pi) = \operatorname{Vol}(C_{\pi'})$ if and only if $\lambda^{\max}(\pi) = \lambda^{\max}(\pi')$.

Next, we need a concrete description of $\lambda_{\max}(\pi)$. Let λ be a partition such that $\lambda_i \leqslant \pi(i)$. In particular,

$$\lambda_1 \leqslant \pi(1),$$
$$\lambda_2 \leqslant \min\{\lambda_1, \pi(2)\} \leqslant \min\{\pi(1), \pi(2)\},$$
$$\vdots$$
$$\lambda_n \leqslant \min\{\lambda_{n-1}, \pi(n)\} \leqslant \min\{\pi(1), \dots, \pi(n)\}.$$

Now, λ is maximal in Young's lattice if we replace all of the above inequalities with equalities. Therefore, $\lambda_i^{\max} = \min\{\pi(1), \dots, \pi(i)\}$.

Recalling Definition 1.1, we now compare $\lambda^{\max}(\pi)$ to $\psi(\pi)$. Observe that the permutation matrix for π has ones in positions $(i, \pi(i))$ and zeros elsewhere, so the ith part of $\psi(\pi)$ is $\min\{\pi(1), \pi(2), \dots, \pi(i)\} - 1$. In other words, one obtains $\psi(\pi)$ by deleting the first column of the Young diagram of $\lambda^{\max}$; this column is necessarily of height n, so one can also reconstruct $\lambda^{\max}(\pi)$ given $\psi(\pi)$ (see Fig. 2, left and center images). We conclude that if π, π' are permutations in S_n, then

$$\mathrm{Vol}(C_\pi) = \mathrm{Vol}(C_{\pi'}) \Leftrightarrow \lambda^{\max}(\pi) = \lambda^{\max}(\pi') \Leftrightarrow \psi(\pi) = \psi(\pi'). \qquad \square$$

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