



Partial Differential Equations/Numerical Analysis

Parallel Schwarz waveform relaxation method for a semilinear heat equation in a cylindrical domain

Méthode de Schwarz pour l'équation de la chaleur non linéaire dans un domaine cylindrique

Minh-Binh Tran

LAGA, UMR 7539 of CNRS, institut Galilée, université Paris XIII, 93430 Villetaneuse, France

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ABSTRACT

We present here a proof of well-posedness and convergence for the parallel Schwarz waveform relaxation algorithm adapted to the semilinear heat equation in a cylindrical domain. It relies on a careful estimate of a local time of existence thanks to the Banach theorem in a well chosen metric space, together with new cylindrical error estimates.

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R É S U M É

Nous proposons dans cette Note une preuve d'existence et de convergence de l'algorithme de Schwarz pour l'équation de la chaleur non linéaire dans un domaine cylindrique. Cette preuve repose sur l'utilisation du théorème de Banach dans un espace bien choisi, et sur de nouvelles estimations d'erreur cylindriques.

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Version française abrégée

L'algorithme de Schwarz sous sa forme moderne fut introduit par P.L. Lions dans [7]. Il proposa également d'utiliser l'algorithme pour des problèmes d'évolution. Cet algorithme, nommé par la suite Schwarz waveform relaxation, par référence aux méthodes de relaxation d'onde [1], fut ensuite étudié indépendamment dans [4] et [5] pour l'équation d'advection-diffusion linéaire. Il consiste à résoudre, sur un intervalle de temps donné, une suite de problèmes de Cauchy avec conditions de Dirichlet hétérogènes dans des sous-domaines qui se recouvrent. Il est bien défini sous certaines conditions de compatibilité. Dans ces articles fondateurs, la convergence de l'algorithme fut établie en utilisant le principe du maximum et la transformation de Laplace en temps.

Une extension à l'équation de réaction-diffusion (2) en dimension 1 fut considérée dans [3]. Pour un problème non linéaire, chaque nouvelle itération de l'algorithme introduit un nouveau temps d'existence plus petit. Avec l'hypothèse $f'(u) \leq C$, ce problème disparaît et les itérées sont définies naturellement sur le même intervalle de temps. La convergence linéaire est ensuite obtenue sur un intervalle de temps non borné par des calculs explicites sur l'équation linéarisée. Une convergence superlinéaire est établie sur un intervalle de temps borné grâce au noyau de la chaleur. Ainsi, l'algorithme hérite des propriétés essentielles de la relaxation d'onde. D'autres non linéarités, en dimension supérieure, furent considé-

E-mail address: binh@math.univ-paris13.fr.

rées par Lui dans [8]. Le cadre de l'étude est la méthode de monotonie dans lequel il n'y a pas non plus de problème de temps d'existence. Notons que dans le cas stationnaire, l'étude avait déjà été faite dans [7].

Nous considérons ici l'équation de la chaleur semi-linéaire (2) dans un domaine cylindrique $\Omega = D \times (a, b)$ de \mathbb{R}^3 , avec une nonlinéarité $f(u)$ de la forme (3). Nous découpons le domaine en bandes cylindriques $\Omega_i = D \times (a_i, b_i)$, avec $a_1 = a$ and $b_I = b$. Ces bandes se recouvrent, c'est-à-dire que pour tout $i \in \llbracket 1, I \rrbracket$, $a_{i+1} < b_i < a_{i+2}$. Dans ces sous-domaines nous résolvons l'équation de réaction–diffusion de manière itérative et en parallèle, voir algorithme (8) et les précisions qui suivent. A chaque étape, nous devons donc résoudre dans Ω_i un problème de Cauchy avec des conditions aux limites de Dirichlet hétérogène. Le domaine n'a pas la régularité voulue pour appliquer les théorèmes classiques, nous avons donc établi de nouvelles preuves. Dans le Théorème 1.1, nous reportons la difficulté dans le second membre de l'équation (5), et nous écrivons un résultat d'existence dans un domaine quelconque. Pour cela, nous introduisons l'équation intégrale associée (10), que nous étudions à l'aide du théorème de Banach dans le sous-espace Y_T de $L^\infty_{loc}((0, T), L^{2p}(\Omega))$, muni de la norme $\|u\|_{Y_T} = \sup_{t \in (0, T)} t^\alpha \|u(t)\|_{2p}$. Nous choisissons une boule B de taille R et liée à la donnée v , et nous pouvons déterminer, à l'aide d'estimations $L^p - L^q$ et du semi-groupe de la chaleur, le temps T_* pour que la fonction Φ définie en (11) soit une stricte contraction de B . Nous avons ainsi une solution unique au problème (10), qui est aussi solution de (5).

Par un relèvement, nous étendons ensuite ce résultat au problème aux limites (7) dans le Théorème 1.2.

Nous pouvons maintenant étudier l'algorithme de Schwarz. Nous commençons par une définition précise de ce que nous entendons par « l'algorithme de Schwarz est bien défini » : il existe un temps T^* tel que pour tout temps $T < T^*$, toutes les itérées vivent sur $(0, T)$ et y sont bornées en norme L^∞ par une même constante.

Le Théorème 1.3 affirme que l'algorithme est bien posé et calcule explicitement le temps T^* . Ce temps dépend des données, et de la taille des sous-domaines. C'est un corollaire du théorème d'existence 1.2, en utilisant le principe du maximum.

Nous montrons enfin au Théorème 1.4 que l'algorithme converge linéairement. Pour démontrer ce résultat, nous établissons des estimations cylindriques. Nous introduisons la fonction $\Psi(x, t) = (u - u_j^k)^2 \exp(\beta(z - a_j) - \gamma t)$, dans Ω_j , et l'opérateur parabolique $\mathcal{L} = \partial_t - \Delta + 2\beta\partial_z$. Pour $\gamma - \beta^2$ suffisamment grand, $\mathcal{L}(\Psi) \leq 0$. Le maximum de Ψ est donc atteint sur l'une des frontières des $\Omega_j \times [0, T)$. Puisque Ψ s'annule au temps initial, et sur les bords extérieurs de Ω , nous obtenons pour t dans $(0, T)$ et x dans D :

$$\begin{aligned} (e_1^k(x, z, t))^2 \exp(\beta(z - a) - \gamma t) &\leq \|(e_2^{k-1}(x, b_1, t))^2 \exp(\beta L_1 - \gamma t)\|_{C(\bar{D} \times [0, T])}, \quad z \in (a, b_1), \\ (e_j^k(x, z, t))^2 \exp(\beta(z - a_j) - \gamma t) &\leq \|(e_{j-1}^{k-1}(x, a_j, t))^2 \exp(-\gamma t)\|_{C(\bar{D} \times [0, T])}, \quad z \in (a_j, b), \\ \forall j, \quad 1 < j < I, \quad \forall z \in (a_j, b_j), \\ (e_j^k(x, z, t))^2 \exp(\beta(z - a_j) - \gamma t) \\ &\leq \max(\|(e_{j+1}^{k-1}(b_j, t))^2 \exp(\beta L_j - \gamma t)\|_{C(\bar{D} \times [0, T])}, \|(e_{j-1}^{k-1}(a_j, t))^2 \exp(-\gamma t)\|_{C(\bar{D} \times [0, T])}). \end{aligned} \tag{1}$$

En parcourant les intervalles de b à a , nous obtenons le théorème. Notons que nous montrons de plus que la convergence est linéaire.

1. Introduction and main results

In the pioneer work [7], P.L. Lions laid the foundations of the modern theory of Schwarz algorithms. He also proposed to use the Schwarz alternating method for evolution equations, and studied the algorithm for nonlinear monotone problems. Later, Schwarz waveform relaxation algorithms were designed independently in [4] and [5] for the linear advection–diffusion equation. Proofs of linear convergence on unbounded time domains, and superlinear convergence on finite time intervals were given in case of n subdomains, using Laplace transform and maximum principle. An extension to the reaction diffusion equation $\partial_t u - \Delta u = f(u)$ in one dimension was considered in [3]. With the assumption $f'(u) \leq C$ in [3], the solution of the nonlinear problem can be compared with the solution of a linear one by the maximum principle, and all iterates are defined naturally on the same time interval. In [8], nonlinearities for which the monotone method applies were considered in general dimension for any number of subdomains. In that case, the sequence of iterates is monotone. In this Note, we consider nonlinearities such that explosion in finite time is permitted, in a cylindrical bounded domain. The difficulty with this kind of nonlinear equations is to have a common time of existence for all iterates of the algorithm. We set explicit formulae for the time of existence and use the maximum principle to obtain convergence.

We consider the semilinear heat equation

$$\partial_t u - \Delta u - f(u) = 0, \tag{2}$$

with the assumptions on f :

$$f \text{ is in } C^1(\mathbb{R}) \text{ and there exists } C_f > 0, p > 1 \text{ such that } |f'(x)| \leq C_f |x|^{p-1} \text{ for all } x \text{ in } \mathbb{R}. \tag{3}$$

We first set an existence theorem for the initial boundary value problem, and more important, new estimates on the solution. We set $p_1 = \frac{3(p-1)}{4p}$, $\alpha = \frac{1}{2}(\frac{1}{p-1} - \frac{3}{4p})$. l_1 and l_2 are positive numbers such that $\frac{1}{l_1} + \frac{1}{l_2} = 1$ and $l_1 p_1 < 1$. We define

$$\tau(r, m) = \left[(4\pi)^{-p_1} \frac{2^{p_1+p\alpha}}{1 - (p_1 + p\alpha)} C_f \max\{1, 2^{p-2}\} (4r + 4m) \right]^{-\frac{8p}{3+p}},$$

$$G(r; T, m_1, m_2) = \left(\frac{(4\pi)^{-p_1 l_1} T^{1-p_1 l_1}}{1 - p_1 l_1} \right)^{-\frac{l_2}{l_1}} \int_0^r [C_f \max\{1, 2^{p-2}\} (m_1 + \zeta^{\frac{p-1}{2}}) \zeta^{\frac{1}{2}} + m_2]^{-l_2} d\zeta. \tag{4}$$

Consider the problem

$$\begin{cases} \partial_t w - \Delta w = f(w + v) & \text{in } \mathcal{O} \times (0, T), \\ w = 0 & \text{on } \partial\mathcal{O} \times [0, T], \\ w(\cdot, 0) = 0 & \text{in } \mathcal{O}. \end{cases} \tag{5}$$

Theorem 1.1. Let \mathcal{O} be a bounded domain in \mathbb{R}^3 , $m(\mathcal{O})$ is its measure. Suppose $v \in C([0, T_0], L^2(\mathcal{O}))$ and $|v| \leq M$ a.e. Denote $R = 2 \max_{|\zeta| \leq M} |f(\zeta)| m(\mathcal{O})^{\frac{1}{2}} \frac{T_0^{(p+3)/4p}}{3(p-1)/4p}$. Then, there exists a local time $T_* = \min(T_0, 1, \tau(R, M^{p-1} m(\mathcal{O})^{\frac{p-1}{2p}}))$, such that for all $T < T_*$, Eq. (5) has a unique solution w in $L^\infty(\mathcal{O} \times (0, T)) \cap C([0, T], L^2(\mathcal{O})) \cap L^2(0, T, H_0^1(\mathcal{O}))$ and $\partial_t w \in L^2(0, T, L^2(\mathcal{O}))$. Furthermore, $\|w\|_{\infty, \infty} \leq M_*$, where

$$M_* := \left(\frac{4T_*}{\pi^3} \right)^{\frac{1}{4}} \left[C_f \max\{1, 2^{p-2}\} \left(G^{-1}(T_*)^{\frac{p-1}{2}} + M^{p-1} m(\mathcal{O})^{\frac{p-1}{2p}} \right) G^{-1}(T_*)^{\frac{p-1}{2}} + m(\mathcal{O})^{\frac{1}{2}} \max_{|\zeta| \leq M} |f(\zeta)| \right],$$

$$G^{-1}(T_*) \equiv G^{-1} \left(T_*; T, M^{p-1} m(\mathcal{O})^{\frac{p-1}{2p}}, m(\mathcal{O})^{\frac{1}{2}} \max_{|\zeta| \leq M} |f(\zeta)| \right). \tag{6}$$

We consider now a bounded cylindrical domain $\Omega = D \times (a, b) \in \mathbb{R}^3$, where D is a bounded domain with smooth boundary ∂D in \mathbb{R}^2 . The boundary $\partial\Omega$ of Ω is made of three parts, $\Gamma_L = \bar{D} \times \{a\}$, $\Gamma_R = \bar{D} \times \{b\}$, and $\Gamma_C = \partial D \times (a, b)$. Dirichlet data g are given on $\partial\Omega \times (0, T)$, defined by g_L on Γ_L , g_R on Γ_R , g_C on Γ_C . These functions are all continuous. We now introduce the basic initial boundary value problem for (2):

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } \Omega \times (0, T), \\ u = g & \text{in } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases} \tag{7}$$

Theorem 1.2. Let u_0 in $C(\bar{\Omega})$ and g in $C(\bar{\partial\Omega})$ with $u_{0|_{\partial\Omega}} = g_{|_{t=0}}$. Let M be a positive constant such that $M > \max(\|u_0\|_\infty, \|g\|_\infty)$. Let R be like in Theorem 1.1. Then, there exists a limit time $T_* = \min(1, \tau(R, M^{p-1} m(\Omega)^{\frac{p-1}{2p}}))$, such that for all $T < T_*$, Eq. (7) has a solution u in $L^\infty(\Omega \times (0, T)) \cap C([0, T], L^2(\Omega)) \cap L^2(0, T, H_0^1(\Omega))$ and $\partial_t u \in L^2(0, T, L^2(\Omega))$. Furthermore $\|u\|_{\infty, \infty} \leq M + M^*$ where M^* is obtained from (6) by replacing \mathcal{O} by Ω .

We divide the domain Ω into I subdomains $\Omega_i = D \times (a_i, b_i)$, with $a_1 = a$ and $b_I = b$. We suppose for each $i \in \llbracket 1, I \rrbracket$ that $a_{i+1} < b_i < a_{i+2}$, we denote by L_i the length of Ω_i : $L_i = b_i - a_i$, and by S_i the size of the overlap $S_i = b_i - a_{i+1}$.

The parallel Schwarz waveform relaxation algorithm solves I equations in I subdomains instead of solving directly the main problem (7). The iterate $\#k$ in the j -th domain, denoted by u_j^k , is defined by

$$\begin{cases} \partial_t u_j^k - \Delta u_j^k = f(u_j^k) & \text{in } \Omega_j \times (0, T), \\ u_j^k(\cdot, a_j, \cdot) = u_{j-1}^{k-1}(\cdot, a_j, \cdot) & \text{in } D \times (0, T), \\ u_j^k(\cdot, b_j, \cdot) = u_{j+1}^{k-1}(\cdot, b_j, \cdot) & \text{in } D \times (0, T), \end{cases} \tag{8}$$

with $u_j^k = g$ on $\partial\Omega_j \cap \partial\Omega \times (0, T)$, $u_j^k(\cdot, \cdot, 0) = u_0$ in Ω_j , and a special treatment for the extreme subdomains, $u_1^k(\cdot, a, \cdot) = g_L$, $u_I^k(\cdot, b, \cdot) = g_R$. An initial guess is provided, i.e. we solve at step 0 Eq. (8), with boundary data on left and right $u_j^0(\cdot, a_j, \cdot) = g_j^0$, $u_j^0(\cdot, b_j, \cdot) = h_j^0$ on $D \times (0, T)$.

Definition 1.1. The parallel Schwarz waveform relaxation algorithm is well-posed if there exists a local time $T^* \leq T_*$ such that for all $T < T^*$, each subproblem (8) in each iteration has a solution over the time interval $(0, T)$, and the set of solutions $\{u_j^k, j \in \llbracket 1, I \rrbracket, k \in \mathbb{N}\}$ is bounded in $C(\bar{\Omega} \times [0, T])$.

Let M a positive number. According to Theorem 1.2, the following problem has a solution φ_M in some interval $C(\bar{\Omega} \times [0, T_0])$:

$$\begin{cases} \partial_t \varphi_M - \Delta \varphi_M = f(\varphi_M) & \text{in } \Omega \times (0, T_0), \\ \varphi_M = M & \text{in } \partial \Omega \times [0, T_0], \\ \varphi_M(\cdot, \cdot, 0) = M & \text{in } \bar{\Omega}. \end{cases} \quad (9)$$

The next theorem gives a common existence time for the iterates:

Theorem 1.3. *Let M be a positive constant such that*

$$M > \max(\|u_0\|_\infty, \|g\|_\infty, (\|g_i^0\|_\infty, \|h_i^0\|_\infty, i \in \llbracket 1, I \rrbracket)).$$

Suppose the data u_0 and g_j, g_j^0 and h_j^0 are continuous and satisfy the compatibility conditions

$$u_0|_{\partial \Omega} = g|_{t=0}, \quad u_0(\cdot, a_j) = g_j^0|_{t=0}, \quad u_0(\cdot, b_j) = h_j^0|_{t=0}.$$

Let M_ be greater than the maximum of φ_M on the boundaries of $\Omega \times (0, T)$ and $\Omega_j \times (0, T)$, $j \in \llbracket 1, I \rrbracket$. Put $T^* = \min(T_0, 1, T_*, \tau(R, M_*^{p-1} m(\Omega_j)^{\frac{p-1}{2p}}), j \in \llbracket 1, I \rrbracket)$. Then the parallel Schwarz waveform relaxation algorithm (8) is well-posed with a local time at least equal to T^* .*

We finally state the convergence of the algorithm:

Theorem 1.4. *With the same assumptions as in Theorem 1.3, let γ be a constant large enough and denote by $\bar{\epsilon}$ the constant $\sqrt{\frac{\gamma}{2} \frac{S_1 \dots S_{I-1}}{L_2 \dots L_{I-1}}}$. If we put $\max_{j \in \llbracket 1, I \rrbracket} \|(u_j^k - u)^2 \exp(-\gamma t)\|_{C(\bar{\Omega} \times (0, T))} = E_k$ and $\bar{E}_k = \max_{j \in \{0, I-1\}} \{E_{k+j}\}$, the parallel Schwarz waveform relaxation algorithm (8) converges linearly: $\bar{E}_n \leq \bar{E}_0 \exp(-n\bar{\epsilon}), \forall n \in \mathbb{N}, T < T^*$ and then $\lim_{n \rightarrow \infty} \max_{j \in \llbracket 1, I \rrbracket} \|u_j^n - u\|_{C(\bar{\Omega}_j \times [0, T])} = 0$.*

2. Problems in the subdomains: The proof of Theorems 1.1 and 1.2

We first prove Theorem 1.1. We introduce the operator $A\zeta = -\Delta\zeta$, with domain $D(A) = \{\zeta \mid \zeta \in H_0^1(\omega), \Delta\zeta \in L^2(\omega)\}$ with its associated Dirichlet semigroup $S(t)$. For every bounded domain \mathcal{O} , we consider the integral equation

$$w(t) = \int_0^t S(t-s)f(w+v)(s) ds. \quad (10)$$

We consider the Banach subspace Y_T of $L_{loc}^\infty((0, T), L^{2p}(\Omega))$, with norm $\|u\|_{Y_T} = \sup_{t \in (0, T)} t^\alpha \|u(t)\|_{2p}$. Let B be the closed ball in Y_T with center 0 and radius R . We introduce the map Φ defined by

$$\Phi(w)(t) = \int_0^t S(t-s)f(w+v)(s) ds. \quad (11)$$

(i) Φ has a fixed-point w in B : The $L^p - L^q$ estimates and Hölder inequality leads us to

$$t^\alpha \int_0^t \|S(t-s)(f(w_1+v) - f(w_2+v))\|_{2p} ds \leq 2\tau(R, T^\alpha \|v\|_{\infty, 2p}) T^{\frac{p+1}{8p}} \|w_1 - w_2\|_{Y_T}. \quad (12)$$

Choose w_2 to be 0, we get that Φ is from B to B . We deduce from the conditions of T that

$$2\tau(T^\alpha \|v\|_{\infty, 2p}) T^{\frac{p+1}{8p}} < (4\pi)^{-\frac{3(p-1)}{4p}} C_f \max\{2^{p-2}, 1\} (2R + 2\|v\|_{\infty, 2p}^{p-1}) \frac{2^{\frac{3(p-1)}{4p} + p\alpha}}{1 - \frac{3(p-1)}{4p} - p\alpha} T_*^{\frac{p+1}{8p}} < 1.$$

Then Φ is a contraction in B for $T \leq T_*$. B is closed in Y_T for its norm. It is therefore a complete metric space, in which we can apply the Banach fixed point theorem.

(ii) By a classical argument, we get $w \in C([0, T], L^2(\Omega))$ and w is also a solution of (5).

(iii) $w \in L^\infty(\Omega \times (0, T))$: Using (10) and Hölder estimates, we get

$$\|w\|_{2p} \leq \frac{T^{1 - \frac{3(p-1)}{4p}}}{1 - \frac{3(p-1)}{4p}} (4\pi)^{-\frac{3(p-1)}{4p}} \left[\int_0^t [C_f \max\{1, 2^{p-2}\} (\|v\|_{\infty, 2p}^{p-1} + \|w\|_{2p}^{p-1}) \|u\|_{2p} + \|f(v)\|_{\infty, 2}]^{1/2} ds \right]^{1/2}.$$

Using Gronwall inequality, we conclude that for any t , $\|w(\cdot, t)\|_{2p} \leq G^{-1}(T_*)^{\frac{1}{2}}$ which proves that $w \in L^\infty(0, T, L^{2p}(\Omega))$. We have now by using the $L^p - L^q$ estimates and Hölder inequality,

$$\|w(t)\|_\infty \leq \pi^{-\frac{3}{4}}(4T_*)^{\frac{1}{4}} [C_f \max\{1, 2^{p-2}\} (G^{-1}(T_*)^{\frac{p-1}{2}} + \|v\|_{\infty, 2p}^{p-1}) G^{-1}(T_*)^{\frac{p-1}{2}} + \|f(v)\|_{\infty, 2}].$$

Theorem 1.2 is now a corollary: we define v as the solution of the linear heat equation in Ω , with zero right-hand side, initial data equal to u_0 , boundary data equal to g . Using the theory in [2], we prove that $v \in \mathcal{C}(\bar{\Omega} \times (0, T)) \cap \mathcal{C}^\infty(\Omega \times (0, T))$. By the maximum principle, $\|v\|_{L^\infty(\Omega \times (0, T))} \leq M$.

3. The Schwarz algorithm: The proof of Theorems 1.3 and 1.4

- (i) Proof of the Theorem of Well-posedness: Due to the maximum principle, $\varphi_M \geq M$ on $\bar{\Omega} \times [0, T_0]$. We prove that the algorithm is well-posed for $T < T^*$ by recursion. Let us describe the first step, and the next steps will use the same process.
 - (a) Using Theorem 1.2, the equation on the j -th subdomain has a solution u_j^1 . Regularity results in [2] and [6], prove that $u_j^1 \in C^{1,2}((0, T) \times \Omega) \cap C(\bar{\Omega}_j \times (0, T))$.
 - (b) Using the Maximum Principle, we prove that $\varphi \geq u_j^1 \geq -M$ on $\bar{\Omega}_j \times (0, T)$.
- (ii) Proof of the Theorem of Convergence: Let $e_j^k = u_j^k - u$. By the previous analysis, the set of e_j^k is bounded in $\mathcal{C}(\bar{\Omega} \times [0, T])$. We call z the coordinate in (a, b) , x the coordinate which varies in D .
 - (a) On the j -th domain, at the k -th iteration, we will estimate e_j^k in $\mathcal{C}(\bar{\Omega} \times [0, T])$. We consider the function $\Psi_j^k(x, z, t) = (e_j^k(x, z, t))^2 \exp(\beta_j(z - a_j) - \gamma t)$, where the coefficients β_j and γ are positive constants to be chosen in the next steps. Define the operator $\mathcal{L}_j = \partial_t - \Delta + 2\beta_j \partial_z$. For $\gamma - \beta_j^2$ large enough, a long calculation shows that $\mathcal{L}_j(\Psi_j^k) \leq 0$ in $\Omega_j \times (0, T)$. Then according to the maximum principle, since the error vanishes on the boundary of $\Omega \times [0, T]$, the maximum of Ψ_j^k can be achieved only on the interfaces $z = a_j$ or $z = b_j$. That implies (1).
 - (b) Running through the intervals starting with the last and the first intervals (a_l, b_l) , (a_1, b_1) , and using the estimates (1), we prove that, for $j = \{2, \dots, l - 1\}$

$$\begin{aligned} (e_{l-j}^{k+j}(x, b_{l-j}, t))^2 \exp(-\gamma t) &\leq \max\{E_k, \dots, E_{k+j-1}\} \exp(-\beta_j S_{l-j}), \\ (e_j^{k+j-1}(x, a_j, t))^2 \exp(-\gamma t) &\leq \max\{E_k, \dots, E_{k+j-2}\} \exp(-\beta'_j S_{j-1}) \end{aligned}$$

with $\beta_j = \beta_1 \frac{S_{l-1}}{L_{l-1}} \dots \frac{S_{l-j+1}}{L_{l-j+1}}$, $\beta'_j = \beta'_1 \frac{S_1}{L_2} \dots \frac{S_{j-1}}{L_j}$. This implies $\bar{E}_{k+1} \leq \bar{E}_k \exp(-\bar{\epsilon})$, $\forall k \in \mathbb{N}$, and we obtain the conclusion of the theorem.

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