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Type-definable groups in C-minimal structures

Groupes type-définissables dans les structures C-minimales

Fares Maalouf

Équipe de logique mathématique, CNRS-UFR de mathématiques, université Paris 7, 175, rue du Chevaleret, 75013 Paris, France

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ABSTRACT

This Note studies type-definable groups in C-minimal structures. We show first for some of these groups, that they contain a cone which is a subgroup. This result will be applied to show that in any geometric locally modular non-trivial C-minimal structure, there is a definable infinite C-minimal group.

RÉSUMÉ

Cette Note traite des groupes type-définissables dans les structures C-minimales. On démontre d'abord pour certains de ces groupes, qu'ils contiennent un cône qui est un sous-groupe. Ce résultat sera appliqué pour montrer que dans toute structure géométrique C-minimale non-triviale et localement modulaire, il y a un goupe C-minimal définissable infini.

1. Introduction

We prove first under certain conditions, that a type-definable group in a C-minimal structure $\mathcal{M}$ contains a cone which is a definable subgroup of $\mathcal{M}$. Similar results are already known in other contexts. In [2], Hrushovski shows that in a stable structure, a type-definable group is the intersections of definable groups. And in the case where the structure is totally transcendental, then the group is in fact definable. More recently, Milliet shows in [6] similar results for *small* theories. A theory is *small*, if for each natural number n, it has countably many n-types over the empty set, and a structure is *small* if its theory is. It is proved in [6] that a $\emptyset$-type-definable group of finite arity in a small structure is the intersection of definable groups, and that for any type-definable group $\mathcal{G}$ in a simple small structure, and any finite subset A of $\mathcal{G}$, there is a definable group containing A.

In Section 2 we prove the following:

Theorem 1. *Let $\mathcal{M} = (M, C, \ldots)$ be a C-minimal structure and $\mathcal{G} = (G, ., 1, C)$ an infinite type-definable C-group in $\mathcal{M}$ such that G is an intersection of cones of M. Then G contains a cone which is a subgroup. In particular, G contains a definable infinite C-group.*

Theorem 1 as well as its proof, are very similar to results which can be found in [5]. In order to be self-contained, we will reproduce here most of the necessary arguments for the proof.

The next result follows from Theorem 1. It can be already found in [5], though it is not stated there as a separate result.

E-mail address: maalouf@logique.jussieu.fr.

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Theorem 2. *Let $\mathcal{G} = (G, ., C, 1, ..)$ be a C-minimal group. Then G contains a cone which is a subgroup.*

Theorem 1 will be used to strengthen a result from [3]. We show the following

Theorem 3. *Let $\mathcal{M} = (M, C, \ldots)$ be a geometric locally modular non-trivial C-minimal structure. Then there is a definable infinite C-minimal group in $\mathcal{M}$.*

Notations: We use $\mathcal{M}, \mathcal{N}, \ldots$ to denote structures and $M, N, \ldots$ for their underlying sets.

We start with a few definitions and preliminary results. C-structures have been introduced and studied in [4,5]. We remind in what follows their definition and principal properties. A C-structure is a structure $\mathcal{M} = (M, C, \ldots)$, where C is a ternary predicate satisfying the following axioms:

- $\forall x, y, z, C(x, y, z) \longrightarrow C(x, z, y)$.
- $\forall x, y, z, C(x, y, z) \longrightarrow \neg C(y, x, z)$.
- $\forall x, y, z, w, C(x, y, z) \longrightarrow C(x, w, z) \bigvee C(w, y, z)$.
- $\forall x, y, x \neq y\ \exists z \neq y, C(x, y, z)$.

Let $\mathcal{M}$ be a C-structure. We call *cone* any subset of M of the form $\{x; \mathcal{M} \models C(a, x, b)\}$, where a and b are two distinct elements of $\mathcal{M}$. It follows from the first three axioms of C-relations that the cones of M form a basis of a completely disconnected topology on M. The last axiom guarantees that all cones are infinite.

Let $(T, \leqslant)$ be a partially ordered set. We say that $(T, \leqslant)$ is a *tree* if the set of elements of T below any fixed element is totally ordered by $\leqslant$, and if any two elements of T have a greatest lower bound. A *branch* of T is a maximal totally ordered subset of T. It is easy to check that if a and b are two distinct branches of T, then $\sup(a \cap b)$ exists. On the set of branches of T, we define a ternary relation C in the following way: we say that $C(a, b, c)$ is true if and only if $b = c$ or a, b, and c are all distinct and $\sup(a \cap b) < \sup(b \cap c)$. It is easy to check that this relation on the set of branches satisfies the first three axioms of a C-relation.

A theorem from [1] shows that C-structures can be looked at as a set of branches of a tree. We will then associate to any C-structure $\mathcal{M}$ a tree T. We will call it the *underlying tree of* $\mathcal{M}$, and the elements of T will be called *nodes*. To any $x, y \in M$, $x \neq y$, we associate the node $t := \sup(x \cap y)$, where x and y are seen as subsets of T. This operation is well defined, and we say then that x and y *branch at* t. If a and b are two elements of M branching at a node t, and if D is the cone $D := \{x \in M; C(a, x, b)\}$, we say then that D is *the cone at the node t containing b*. If t and t' are two nodes, we denote by $t \,||\, t'$ the property that t and t' are not comparable in T with respect to the relation $\leqslant$. If A and B are two sets of nodes, we denote by $A \,||\, B$ the property that, for any $t \in A$ and $t' \in B, t \,||\, t'$.

Definition 4. Let $\mathcal{M} = (M, C, \ldots)$ be a C-structure. We say that $\mathcal{M}$ is *C-minimal* if and only if for any structure $\mathcal{M}' = (M', C, \ldots)$ elementarily equivalent to $\mathcal{M}$, any definable subset of M' can be defined without quantifiers using only the relations C and $=$.

2. A cone of a C-minimal group is a subgroup

Definition 5. We say that $\mathcal{G} = (G, ., 1, C)$ is a C-group if and only if $\mathcal{G}$ is a C-structure, $(G, ., 1)$ is a group and for all $x, y, z, a, b \in G, \mathcal{G} \models C(x, y, z) \longrightarrow C(a.x.b, a.y.b, a.z.b)$.

Let $\mathcal{G} = (G, ., 1, C)$ be a C-group and T its underlying tree. Let $t \in T$, and $x, y, x', y', z \in G$ be such that, x and y, as well as x' and y' branch at t (recall that the elements of a C-structure are looked at as branches of the underlying tree). It is easy to check that $z.x$ and $z.y$, as well as $z.x'$ and $z.y'$, all branch again at the same node, which we denote by t^z. We can then define a left action of G on T, $(z, t) \mapsto t^z$, and check that this action preserves $<$. We will speak then of *orbits of G on T*. Similarly one can check that if D is a cone of G at a node t, then $z.D := \{z.x; x \in D\}$ is a cone at the node t^z.

Proposition 6. *Let $\mathcal{G} = (G, ., 1, C)$ be a C-group and T its underlying tree. Suppose that some orbit Ω of G on T is an antichain. Then there is a cone in G which is a subgroup.*

Proof. Let $s \in \Omega$ and $g \in G$ be such that $s \in g$. Since $g^{-1}.g = 1$, then $t := s^{g^{-1}}$ is an element of $\Omega \cap 1$ (here we see 1 as a branch of T). Let X be the cone at the node t containing 1. We want to show that X is a subgroup of G. Take $h \in X$. Then $h.X$ is a cone at the node t^h. But $t^h \in \Omega$, and if $t^h \neq t$, t^h is incomparable with t (Ω is an antichain). In this case, since a chain contains no two incomparable elements, $h.X \cap X = \emptyset$. But this cannot happen since 1 and h are two elements of X, and thus $h \in h.X \cap X$. We have shown that $h.X$ is the cone at the node t containing h, and then $h.X = X$. And since $1 \in X$, $h^{-1} \in X$. Since this is true for any $h \in X$, X is a subgroup of G. □

We now will show in what follows that the same result holds for the C-groups of the statement of Theorem 1 in the case where there is an orbit which is not an antichain.

Lemma 7. *Let $\mathcal{M} = (M, C, \ldots)$ and $\mathcal{G} = (G, ., 1, C)$ be as in the statement of Theorem 1. There is a definable subset V of M and an $\mathcal{M}$-definable function $F : V \times V \longmapsto M$ such that $G \subset V$, $F|G \times G = .$ and F is a C-isomorphism in each variable.*

Proof. By compactness we know that the group operation of $\mathcal{G}$ is definable in $\mathcal{M}$. Denote then by F an $\mathcal{M}$-definable ternary relation which restriction to G is ".", the group operation of $\mathcal{G}$. For an element x of M, denote by $C_x := \{y \in M; \neg C(y, x, 1)\}$. Let V be the set of elements x of M such that, F defines on $C_x \times C_x$ a binary function which is a C-isomorphism in each variable. So V is definable, and using the fact that G is an intersection of cones of M, we see easily that V and F do the job. □

Notations: Let from now on $\mathcal{M} = (M, C, \ldots)$ and $\mathcal{G} = (G, ., 1, C)$ be as in the statement of Theorem 1, and let V and F be as in the statement of Lemma 7. $(T, <)$ will be the underlying tree of $\mathcal{G}$, and $(T', <)$ will be the underlying tree of V (we have that $T \subset T'$).

Now doing as above, but using F instead of ., we can define the left action of V on T'. We will speak then of *orbits of V on T' via F*.

Lemma 8. *Let $t \in T$ and $x \in V \setminus G$. Then $t^x \notin T$.*

Proof. Since G is an intersection of cones of M, it is enough to show that for all $y \in G$, $F(x, y) \notin G$. Suppose not. If $F(x, y) = z \in G$, by the fact that F is one-to-one in each variable and the fact that the restriction of F to $G \times G$ is the operation "." of G, we get that $x = y^{-1}.z \in G$. Contradiction. □

Lemma 9. *Let Ω be an orbit of G on T and $z \in G$. Then in T, $\Omega \cap z$ is a finite union of intervals and points.*

Proof. Let $t \in \Omega$, and Ω' be the orbit of V on T' via F containing t. By C-minimality $\Omega' \cap z$ is a finite union of intervals and points. By Lemma 8, $\Omega = \Omega' \cap T$, so $\Omega \cap z = \Omega' \cap z \cap T$. And then in T, $\Omega \cap z$ is a finite union of intervals and points. □

The two following results can be found in [5]. But we will restate them here in a slightly more general context.

Proposition 10. *Let Ω be an orbit of G on T. Then there are no elements $r, s, t \in \Omega$ such that $r < s$, $r < t$ and $s \,||\, t$.*

Proof. This is exactly Lemma 4.6 of [5], except for the fact that in our case $\mathcal{G}$ is not necessarily C-minimal: $\mathcal{G}$ satisfies only the hypothesis of Theorem 1, namely that it is a type-definable C-group in a C-minimal structure $\mathcal{M}$ and its universe G is an intersection of cones of $\mathcal{M}$. The same proof of Lemma 4.6 of [5] works as well in our case, except for replacing the centralizer of an element h of $\mathcal{G}$ by the set $C'_G(h) := \{x \in V, F(x, h) = F(h, x)\}$. Lemma 7 is used only at this step. □

Let Ω be an orbit of G on T. For all $t \in \Omega$, let $L_t := \{t' \in \Omega; t \leqslant t' \vee t \geqslant t'\}$. Using Proposition 10, it is easy to check that the relation $\sim$ defined on Ω by $t \sim t' \longleftrightarrow t' \in L_t$ is an equivalence relation. For all $t, t' \in \Omega$, we denote by $\bar{t}$ the class of t modulo $\sim$. Note that $\bar{t} = \bar{t'}$ if and only if $L_t = L_{t'}$. Set $L_{\bar{t}} := L_t$. It is obvious that $\{L_{\bar{t}}; \bar{t} \in \Omega/\sim\}$ is a partition of Ω and that if $\epsilon \neq \epsilon' \in \Omega/\sim$, $L_\epsilon \,||\, L_{\epsilon'}$ (for the notations see the introduction). Now if Ω is not an antichain, at least one of the L_ϵ is not a singleton. And since G acts transitively on the set $\{L_{\bar{t}}; \bar{t} \in \Omega/\sim\}$, none of the L_ϵ is a singleton.

Proposition 11. *Let Ω be an orbit of G on T which is not an antichain. As above, we write $\Omega := \dot{\bigcup}_{\epsilon \in \Omega/\sim} L_\epsilon$. Then for all ϵ, there is no $g \in G$ such that $L_\epsilon \subset g$.*

Proof. Suppose for a contradiction that for some ϵ, g, $L_\epsilon \subset g$. Let $s < t \in L_\epsilon$, and $h \in G$ be such that h branches with g in s. The image $L_\epsilon^{g^{-1}.h}$ of L_ϵ under the left action of $g^{-1}.h$ is a subset of h. But from Proposition 10 and the fact that $\Omega^{g^{-1}.h} = \Omega$, $L_\epsilon^{g^{-1}.h}$ contains no elements of h above s, we get that $L_\epsilon^{g^{-1}.h} \subset g \cap h$. On the other hand, we can find $t_1 \in \Omega$ such that $t_1^{g^{-1}.h} = t$. But then $t_1 \notin L_\epsilon$ and t_1 is not comparable with t. But $t_1^{g^{-1}.h} \in L_\epsilon$ is comparable with $t^{g^{-1}.h} \in h \cap g$. Contradiction. □

Proposition 12. *Suppose that some orbit Ω of G on T is not an antichain. Then there is a cone of $\mathcal{G}$ which is a C-subgroup.*

Proof. We use the notations of Proposition 11. Let L_ϵ be such that $1 \cap L_\epsilon \neq \emptyset$. Let x be an element of G containing a node of $L_\epsilon \setminus 1$ (such an element exists by Proposition 11). Let t be the node of T at which x branches with 1. We want to show that the cone containing 1 at the node t is a subgroup of G. Denote this cone by D, and let $h \in D$. Since $1 \in D$, $h.D$ is a cone containing h at the node t^h. Note first that $t, t^h \in h$, then either $t \leqslant t^h$ or $t^h \leqslant t$. Note also that $t^h \in L_\epsilon$. We want to show that $h.D = D$. Since $h \in D \cap h.D$, it is enough to show that $t^h = t$. Suppose not. If $t^h > t$, then by the definitions of t and D and the fact that $t^h \in L_\epsilon$, $t^h \notin h$. But this is impossible. And if $t^h < t$, then $t^{h^{-1}} > t$, and for the same reason as above, $1 \notin h^{-1}.D$, which is impossible. So for all $h \in D$, $h.D = D$, and since $1 \in D$, D is a C-subgroup of $\mathcal{G}$. □

Theorem 1 follows directly from Propositions 6 and 12.

Proof of Theorem 3. Let $\mathcal{M} = (M, C, \ldots)$ be a non-trivial locally modular geometric C-minimal structure, and let $\mathcal{M}'$ be an ω_1-saturated structure elementarily equivalent to $\mathcal{M}$. We show in [3] that in $\mathcal{M}'$ there is an infinite type-definable C-group $\mathcal{G}' = (G', ., 1, C)$, and moreover, G' is an intersection of cones of M. Thus $\mathcal{G}'$ satisfies the hypothesis of Theorem 1, and there is a cone D of G' which is an infinite C-group definable in $\mathcal{M}'$. Since $\mathcal{M} \equiv \mathcal{M}'$, there is an infinite C-group $\mathcal{G}$ definable in $\mathcal{M}$. And $\mathcal{G}$ is C-minimal because $\mathcal{M}$ is C-minimal. □

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