C. R. Acad. Sci. Paris, Ser. I 348 (2010) 873–877

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

Liouville-type theorems for certain degenerate and singular parabolic equations

Théorèmes de type Liouville pour quelques équations paraboliques singulières dégénérées

Emmanuele DiBenedetto[a], Ugo Gianazza[b], Vincenzo Vespri[c]

[a] *Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA*
[b] *Dipartimento di Matematica "F. Casorati", Università di Pavia, via Ferrata 1, 27100 Pavia, Italy*
[c] *Dipartimento di Matematica "U. Dini", Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy*

A R T I C L E I N F O

Article history:
Received 5 April 2010
Accepted 17 June 2010
Available online 23 July 2010

Presented by Pierre-Louis Lions

A B S T R A C T

Relying on recent results on Harnack inequalities for equations of p-Laplacian type, we prove Liouville-type estimates for solutions to these equations, both in the degenerate ($p > 2$), and in the singular ($1 < p < 2$) range.

R É S U M É

En utilisant des résultats récents sur l'inégalité de Harnack pour les équations type p-laplacien, on établit des théorèmes de type Liouville pour les solutions de ces équations, dans le cas dégénéré $p > 2$, ainsi bien que dans le cas singulier $1 < p < 2$.

Version française abrégée

On sait que, pour les solutions de l'équation de la chaleur, des limitations unilatérales ne sont pas suffisantes pour garantir qu'elles sont constantes.

Des résultats analogues sont valables pour les solutions faibles des équations (1), (2) dans le cas $p > 2$.

Le résultat fondamental de cette Note montre que, dans l'intervalle singulier sur-critique (3), les solutions faibles de (1), (2), définies dans tout $\mathbb{R}^{N+1}$ et bornées inférieurment (ou supérieurment) sont en fait constantes (Théorème 1.2).

Ce théorème n'est plus vrai quand p est dans l'intervalle singulier critique et sous-critique (4), comme on peut voir, grâce à certaines solutions explicites de $(1)'$ dans cet intervalle.

Dans le cas dégénéré ($p > 2$) il est nécessaire de supposer des limitations soit inférieures soit supérieures, pour pouvoir garantir que la solution est constante. Il est possible de formuler ces limitations bilatérales de différentes façons, comme on le montre dans le Théorème 1.1 et dans les Propositions 1.1, 1.2.

1. Liouville-type theorems

For $T \in \mathbb{R}$ let S_T denote the semi-infinite strip

$$S_T = \mathbb{R}^N \times (-\infty, T).$$

E-mail addresses: em.diben@vanderbilt.edu (E. DiBenedetto), gianazza@imati.cnr.it (U. Gianazza), vespri@math.unifi.it (V. Vespri).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.06.019

Let u be a non-negative, local, weak solution to the quasi-linear parabolic equation

$$\begin{aligned} &u \in C_{\mathrm{loc}}\big(-\infty, T; L^2_{\mathrm{loc}}(\mathbb{R}^N)\big) \cap L^p_{\mathrm{loc}}\big(-\infty, T; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)\big),\\ &u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0 \quad \text{weakly in } S_T, \end{aligned} \tag{1}$$

for $p > 1$, where $\mathbf{A}: S_T \to \mathbb{R}^N$, is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du) \cdot Du \geqslant C_o |Du|^p \\ \big|\mathbf{A}(x,t,u,Du)\big| \leqslant C_1 |Du|^{p-1} \end{cases} \quad \text{a.e. in } S_T \tag{2}$$

where C_o and C_1 are given positive constants. The prototype is

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0, \quad \text{in } S_T. \tag{1$'$}$$

The modulus of ellipticity of this class of equation is $|Du|^{p-2}$ and accordingly they are degenerate for $p > 2$ and singular for $1 < p < 2$.

Harmonic functions in $\mathbb{R}^N$ with one-sided bound, are constant. This, known as the Liouville theorem, is solely a consequence of the Harnack inequality. As such it extends to solutions to homogeneous, quasi-linear, elliptic partial differential equations in $\mathbb{R}^N$ with one-sided bound.

This property does not extend to caloric functions in $\mathbb{R}^N \times \mathbb{R}$, as a one-sided bound is not sufficient to imply that they are constant. The function

$$\mathbb{R} \times \mathbb{R} \ni (x,t) \to u(x,t) = e^{x+t}$$

is a non-negative, non-constant solution of the heat equation in $\mathbb{R} \times \mathbb{R}$. The Liouville theorem continues to be false for non-negative solutions to degenerate p-Laplacian type equations ($p > 2$). The one-parameter family of non-negative functions defined in the whole $\mathbb{R} \times \mathbb{R}$

$$u(x,t;c) = A(1 - x + ct)_+^{\frac{p-1}{p-2}}, \quad \text{where } A = c^{\frac{1}{p-2}} \left(\frac{p-2}{p-1}\right)^{\frac{p-1}{p-2}}$$

is a non-negative, non-constant, weak solution to (1)$'$ in $\mathbb{R}^2$.

The main result of this note is that the Liouville property while false for p in the degenerate range $p > 2$, it does actually holds true for p in the singular, super-critical range

$$\frac{2N}{N+1} < p < 2 \tag{3}$$

and then it is false again for p in the singular, critical, and sub-critical range

$$1 < p \leqslant \frac{2N}{N+1}. \tag{4}$$

While some results appear in the literature for linear and *coercive* equations ($p = 2$) (see for example [4–6]), to our knowledge, no results are known for degenerate ($p > 2$) or singular ($1 < p < 2$) quasi-linear equations of the type of (1)–(2).

1.1. Two-sided bounds and Liouville-type theorems in the degenerate range $p > 2$

Henceforth we let u be a continuous, local, weak solution to (1)–(2) in S_T for $p > 2$.

Theorem 1.1. *If u is bounded in S_T, then u is constant.*

The next proposition asserts that if a one-sided bound is available, then it suffices to verify the two-sided bound only at some time level.

Proposition 1.1. *Let u be bounded below in S_T and assume that*

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s < +\infty \quad \textit{for some } s < T.$$

Then u is constant in S_s.

It has been observed that a one-sided bound on u is not sufficient to infer that u is constant in S_T. Such a conclusion however holds if u has a two-sided bound as indicated by Theorem 1.1. Consider the family of functions

$$u(x,t) = C(N,p)\left(\frac{|x|^p}{T-t}\right)^{\frac{1}{p-2}}$$

defined in S_T, where

$$C(N,p)=\left[\frac{1}{\lambda}\left(\frac{p-2}{p}\right)^{p-1}\right]^{\frac{1}{p-2}} \quad \text{and} \quad \lambda=N(p-2)+p.$$

One verifies that this solves the prototype equation $(1)'$ in S_T, for any $p>2$, and it blows up as $t\to T$, for all $x\in\mathbb{R}^N-\{0\}$. This suggests that if u is defined in the whole $\mathbb{R}^N\times\mathbb{R}$, a condition weaker than a two-sided bound might imply that u is constant. The next proposition is in this direction; it asserts that it suffices to check the two-sided boundedness of u at a single point $y\in\mathbb{R}^N$, for large times, to conclude that u is constant.

Proposition 1.2. *Let u be defined and bounded below in $\mathbb{R}^N\times\mathbb{R}$. If*

$$\lim_{s\to+\infty} u(y,s)=\alpha \quad \text{for some } y\in\mathbb{R}^N \text{ and some } \alpha\in\mathbb{R},$$

then u is constant.

1.2. One-sided bounds and Liouville-type theorems in the singular super-critical range (3)

Theorem 1.2. *Let u be a continuous, local, weak solution to the singular, quasi-linear equation* (1)–(2) *in $\mathbb{R}^N\times\mathbb{R}$, for p in the singular, super-critical range* (3)*. If u has a one-sided bound, then it is constant.*

The theorem is false for p in the singular, critical and sub-critical range (4). Consider the two-parameter family of functions

$$u(x,t)=(T-t)_+^{\frac{N+2}{4}}\left(a+b|x|^{\frac{2N}{N-2}}\right)^{-\frac{N}{2}},$$
$$N>2, \qquad p=\frac{2N}{N+2}<\frac{2N}{N+1}$$

where $a>0$ and T are parameters, and

$$b=b(N,a)=\frac{N-2}{N^2}\left(\frac{N+2}{4Na}\right)^{\frac{N+2}{N-2}}.$$

They are non-negative, non-constant, locally bounded, weak solutions to the prototype p-Laplacian equation $(1)'$ in $\mathbb{R}^N\times\mathbb{R}$. For the critical value $\frac{2N}{N+1}$ the function

$$u(x,t)=\left(|x|^{\frac{2N}{N-1}}+e^{bt}\right)^{-\frac{N-1}{2}},$$
$$b=\frac{2N^{\frac{2N}{N+1}}}{N-1}, \qquad N\geqslant 2, \qquad p=\frac{2N}{N+1}$$

is a non-negative, non-constant solution to $(1)'$ in $\mathbb{R}^N\times\mathbb{R}$.

2. Intrinsic Harnack estimates [1,3]

Let u be a continuous, non-negative, local, weak solution to (1)–(2). Fix $(x_o,t_o)\in S_T$ such that $u(x_o,t_o)>0$ and construct the cylinders

$$(x_o,t_o)+Q_\rho^\pm(\theta)=B_\rho(x_o)\times\left(t_o\pm\theta\rho^p\right) \tag{5}$$

where $B_\rho(x_o)$ is the ball in $\mathbb{R}^N$ centered at x_o and of radius ρ, and

$$\theta=\delta u(x_o,t_o)^{2-p} \tag{6}$$

for a constant $\delta>0$. These cylinders are "intrinsic" to the solution since their height is determined by the value of u at (x_o,t_o). The point (x_o,t_o) and the constant δ being determined, we let $\rho>0$ be so that

$$(x_o,t_o)+Q_{8\rho}^\pm(\theta)\subset S_T. \tag{7}$$

Theorem 2.1. *(See [1,2].) Let u be a continuous, non-negative, local, weak solution to the degenerate equations* (1)–(2) *in S_T, for $p>2$. There exist constants $\delta,\gamma>1$ depending only upon the data $\{p,N,C_o,C_1\}$, such that for all intrinsic cylindersv as in* (5)–(7)*, there* *holds*

$$\gamma^{-1}\sup_{B_\rho(x_o)} u\big(\cdot, t_o-\theta\rho^p\big) \leqslant u(x_o,t_o) \leqslant \gamma \inf_{B_\rho(x_o)} u\big(\cdot, t_o+\theta\rho^p\big). \tag{8}$$

The constants γ and δ deteriorate as $p \to \infty$, but they are "stable" as $p \to 2$. Thus by formally letting $p \to 2$ in (8) one recovers the classical Moser's Harnack inequality [7].

Theorem 2.2. *(See [3,2].) Let u be a continuous, non-negative, local, weak solution to the singular equations* (1)–(2)*, in S_T, for p in the super-critical range* (3)*. There exist constants $\delta \in (0,1)$ and $\gamma > 1$, depending only upon the data $\{p, N, C_o, C_1\}$, such that for all intrinsic cylinders as in* (5)–(7)*, there holds*

$$\gamma^{-1}\sup_{B_\rho(x_o)} u(\cdot,\sigma) \leqslant u(x_o,t_o) \leqslant \gamma \inf_{B_\rho(x_o)} u(\cdot,\tau) \tag{9}$$

for any pair of time levels σ, τ in the range

$$t_o - \delta u(x_o,t_o)^{2-p}\rho^p \leqslant \sigma, \quad \tau \leqslant t_o + \delta u(x_o,t_o)^{2-p}\rho^p. \tag{10}$$

The constants δ and γ^{-1} tend to zero as either $p \to 2$ or $p \to \frac{2N}{N+1}$.

Both, right and left inequalities in (9) are insensitive to the times σ, τ, provided they range within the time-intrinsic geometry of (5)–(7). For $\sigma = \tau = t_o$ the theorem yields

Corollary 2.1 *(The Elliptic Harnack Inequality [3]). Let u be a continuous, non-negative, local, weak solution to the singular equations* (1)–(2) *for p in the super-critical range* (3)*. Then for all intrinsic cylinders as in* (5)–(7)*, there holds*

$$\gamma^{-1}\sup_{B_\rho(x_o)} u(\cdot,t_o) \leqslant u(x_o,t_o) \leqslant \gamma \inf_{B_\rho(x_o)} u(\cdot,t_o). \tag{11}$$

The right and left inequalities in (9) are simultaneously forward, backward and elliptic Harnack estimates. Inequalities of this type are false for non-negative solutions to the heat equation [7]. This is reflected in that the constants δ and γ^{-1} tend to zero as $p \to 2$. These inequalities loose meaning also as p tends to the critical value $\frac{2N}{N+1}$. The range (3) of p is optimal for Theorem 2.2 and Corollary 2.1 to hold [3].

3. Proofs of the Liouville-type statements

Assume $p > 2$. If u is bounded above (below) in S_T set

$$M = \sup_{S_T} u \quad \Big(m = \inf_{S_T} u\Big)$$

and for points $(y,s) \in S_T$ for which $M > u(y,s)$, $(u(y,s) > m$ respectively) construct the intrinsic, backward p-paraboloid(s)

$$P_M(y,s) = \big\{(x,t) \in S_T \,|\, t-s \leqslant -\delta\big[M-u(y,s)\big]^{2-p}|x-y|^p\big\}$$
$$\big(P_m(y,s) = \big\{(x,t) \in S_T \,|\, t-s \leqslant -\delta\big[u(y,s)-m\big]^{2-p}|x-y|^p\big\}\big),$$

where δ is the constant in the intrinsic Harnack inequality of Theorem 2.1. The proof of Theorem 1.1 is an immediate consequence of the following:

Lemma 3.1. *Let u be bounded below* (*above*) *in S_T. Then for all $x \in \mathbb{R}^N$*

$$\lim_{t\to-\infty} u(x,t) = \inf_{S_T} u \qquad \Big(\lim_{t\to-\infty} u(x,t) = \sup_{S_T} u\Big)$$

and the limit is uniform in any p-paraboloid $P_m(y,s)$ ($P_M(y,s)$ respectively).

Proof. Having fixed $\varepsilon > 0$, there exists $(y_\varepsilon, s_\varepsilon) \in S_T$, such that

$$u(x_\varepsilon, t_\varepsilon) - m = \frac{\varepsilon}{\gamma}$$

where γ is the constant in the intrinsic, backward Harnack inequality in (8). Applying such inequality to $(u-m)$, gives

$$m \leqslant u(y,s) \leqslant m + \varepsilon, \quad \text{for all } (y,s) \in P_m(y_\varepsilon, s_\varepsilon).$$

Now, for all fixed $x \in \mathbb{R}^N$, the half-line $[t < T] \times \{x\}$ enters the p-paraboloid $P_m(y_\varepsilon, s_\varepsilon)$ for some t. □

Proof of Proposition 1.1. May assume $m = 0$. The assumption implies

$$0 \leqslant u(y,s) \leqslant M_s < \infty \quad \text{for all } y \in \mathbb{R}^N.$$

By the backward, intrinsic Harnack inequality (8)

$$0 \leqslant u \leqslant \gamma M_s \quad \text{in } P_m(y,s) \text{ for all } y \in \mathbb{R}^N.$$

Hence $0 \leqslant u \leqslant \gamma M_s$ in S_s, and by Theorem 1.1 u is constant in S_s. □

Proof of Proposition 1.2. Assume $m = 0$, and $\alpha > 0$. There exists a sequence $\{s_n\} \to \infty$, such that for all arbitrary but fixed $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\alpha - \varepsilon < u(y, s_n) < \alpha + \varepsilon, \quad \text{for all } n \geqslant n_\varepsilon.$$

Fix $s > s_{n_\varepsilon}$, and define a sequence of radii $\{\rho_n\}$, such that

$$s_n - \left(\frac{c}{\alpha+\varepsilon}\right)^{p-2} \rho_n^p = s \quad \Rightarrow \quad \rho_n = \left[(s_n - s)\left(\frac{\alpha+\varepsilon}{c}\right)^{p-2}\right]^{\frac{1}{p}}.$$

By the intrinsic, backward Harnack inequality in (8)

$$\sup_{B_{\rho_n}} u\left(\cdot, s_n - \left(\frac{c}{u(y,s_n)}\right)^{p-2} \rho_n^p\right) \leqslant \gamma u(y,s_n) \leqslant \gamma(\alpha+\varepsilon)$$

which we rewrite as

$$\sup_{B_{\rho_n}} u(\cdot, s) \leqslant \gamma(\alpha + \epsilon).$$

Now let $n \to \infty$ by keeping $s > s_{n_\varepsilon}$ fixed. Then $\rho_n \to \infty$ and the previous inequality implies

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s \leqslant \gamma(\alpha + \varepsilon).$$

The conclusion follows from Proposition 1.1, since $s > s_{n_\varepsilon}$ is arbitrary. □

Remark 3.1. Assuming $\alpha > 0$ for simplicity, the same argument continues to hold, if there exists a sequence $\{(y_n, s_n)\} \subset \mathbb{R}^N \times \mathbb{R}$ and $s \in \mathbb{R}$, such that $s_n \to +\infty$,

$$s_n - s = \left(\frac{c}{\alpha}\right)^{p-2} |y_n|^p,$$

and $\lim_{n\to+\infty} u(y_n, s_n) = \alpha$.

Proof of Theorem 1.2. It suffices to assume that u is non-negative and non-constant. Fix $(x_o, t_o) \in \mathbb{R}^N \times \mathbb{R}$ such that $u(x_o, t_o) > 0$. By the Harnack inequality (11), for any $\rho > 0$,

$$u(x_o, t_o) \leqslant \gamma \inf_{B_\rho(x_o)} u(\cdot, t_o).$$

Now let $\rho \to +\infty$ and deduce that $u(x, t_o) = 0$ for all $x \in \mathbb{R}^N$. The left-hand side, intrinsic Harnack inequality (9)–(10) now implies that $u \equiv 0$. □

References

[1] E. DiBenedetto, U. Gianazza, V. Vespri, Harnack estimates for quasi-linear degenerate parabolic differential equation, Acta Math. 200 (2008) 181–209.

[2] E. DiBenedetto, U. Gianazza, V. Vespri, Alternative forms of the Harnack inequality for non-negative solutions to certain degenerate and singular parabolic equations, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. (9) Rend. Lincei Mat. Appl. 20 (2009) 369–377.

[3] E. DiBenedetto, U. Gianazza, V. Vespri, Forward, backward and elliptic Harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) IX (2010) 385–422.

[4] R.Ya. Glagoleva, Liouville theorems for the solution of second order linear parabolic equations with discontinuous coefficients, Math. Zametki 5 (1969) 599–606.

[5] R.Ya. Glagoleva, Phragmén–Lindelöf type theorems and Liouville theorems for a linear parabolic equation, Math. Zametki 37 (1985) 119–124.

[6] A.E. Kogoj, E. Lanconelli, Liouville theorems in halfspaces for parabolic hypoelliptic equations, Ric. Mat. 55 (2006) 267–282.

[7] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17 (1964) 101–134.