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Topology

On the commutator length of a Dehn twist

*Sur la longueur des commutateurs d'un twist de Dehn*Błażej Szepietowski¹

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ABSTRACT

We show that on a nonorientable surface of genus at least 7 any power of a Dehn twist is equal to a single commutator in the mapping class group and the same is true, under additional assumptions, for the twist subgroup, and also for the extended mapping class group of an orientable surface of genus at least 3.

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R É S U M É

Nous démontrons que sur une surface non orientable de genre au moins 7 toute puissance d'un twist de Dehn est égale à un unique commutateur dans le groupe de difféotopies et que ceci est vrai, sous conditions additionnelles, pour le sous-groupe généré par les twists, aussi bien que pour l'extension du groupe de difféotopies d'une surface orientable de genre au moins 3.

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1. Introduction

Let S be a closed surface of genus g . If S is nonorientable, then g is the number of projective planes in a connected sum decomposition. The *mapping class group* $\mathcal{M}(S)$ of S is the group of isotopy classes of all, orientation preserving if S is orientable, self-homeomorphisms of S . For orientable S , the *extended mapping class group* $\mathcal{M}^\circ(S)$ is the group of isotopy classes of all self-homeomorphisms of S , including those reversing orientation. For notational convenience we define $\mathcal{M}^\circ(S)$ to equal $\mathcal{M}(S)$ for nonorientable S .

For a two-sided simple closed curve c on S we denote by t_c a Dehn twist about c . For a Dehn twist t_c we always assume that c does not bound a disc or a Möbius band, so that t_c is a nontrivial element of $\mathcal{M}(S)$. It is well known that $\mathcal{M}(S)$ is generated by Dehn twists if S is orientable. If S is nonorientable, then the *twist subgroup* $\mathcal{T}(S)$ generated by all Dehn twists has index 2 in $\mathcal{M}(S)$ (cf. [9]).

For a group G let $[G, G]$ denote the commutator subgroup generated by all commutators $[a, b] = aba^{-1}b^{-1}$. For $x \in [G, G]$ the *commutator length* $cl_G(x)$ is the minimum number of factors needed to express x as a product of commutators. The *stable commutator length* is the limit

$$scl_G(x) = \lim_{n \rightarrow \infty} \frac{cl_G(x^n)}{n}.$$

Recall that the first homology group $H_1(G; \mathbb{Z})$ of G is isomorphic to the quotient $G/[G, G]$.

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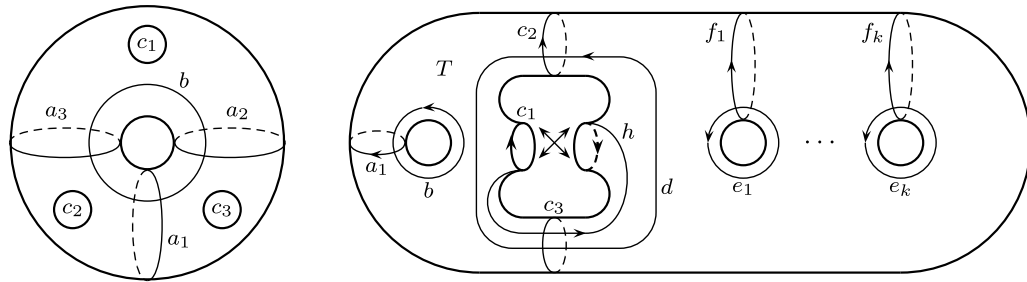


Fig. 1. The torus T and its embedding in a closed nonorientable surface S such that $S \setminus c_1$ is orientable.

Fig. 1. Le tore T et son plongement dans une surface fermée non orientable S , telle que $S \setminus c_1$ est orientable.

For orientable S and $g \geq 3$ it is well known that $\mathcal{M}(S)$ is perfect, i.e. $\mathcal{M}(S) = [\mathcal{M}(S), \mathcal{M}(S)]$ (cf. [10]), and for any Dehn twist t_c we have $cl_{\mathcal{M}(S)}(t_c) = 2$ and $scl_{\mathcal{M}(S)}(t_c) > 0$ [1,2,5,6]. For nonorientable S the groups $H_1(\mathcal{M}(S); \mathbb{Z})$ and $H_1(\mathcal{T}(S); \mathbb{Z})$ were computed by Korkmaz [4] and Stukow [11]. In particular, if $g \geq 7$ then we have $[\mathcal{M}(S), \mathcal{M}(S)] = [\mathcal{T}(S), \mathcal{T}(S)] = \mathcal{T}(S)$. In this Note we prove the following:

Theorem 1.1. *Let S be a closed orientable surface of genus $g \geq 3$ or a closed nonorientable surface of genus $g \geq 7$. Then for every two-sided simple closed curve c on S and every $n \in \mathbb{Z}$, t_c^n is equal to a single commutator of elements of $\mathcal{M}^\circ(S)$.*

For even n Theorem 1.1 and hence also Corollary 1.3 below, follow immediately from the fact that t_c is conjugate to its inverse in $\mathcal{M}^\circ(S)$ (see Remark below), and were known already, at least for orientable S [7, Remark 12].

Theorem 1.2. *Let c be a two-sided simple closed curve on a closed nonorientable surface S satisfying one of the following assumptions:*

- c is separating and $g \geq 7$, or
- $S \setminus c$ is connected and nonorientable and $g \geq 8$, or
- $S \setminus c$ is connected and orientable, $g \geq 6$ and $g \equiv 2 \pmod{4}$.

Then for any $n \in \mathbb{Z}$, t_c^n is equal to a single commutator of elements of $\mathcal{T}(S)$.

The following corollary is an immediate consequence of Theorems 1.1 and 1.2 and the definition of the stable commutator length:

Corollary 1.3. *For S and c as in Theorem 1.1 or Theorem 1.2 we have respectively $scl_{\mathcal{M}^\circ(S)}(t_c) = 0$ or $scl_{\mathcal{T}(S)}(t_c) = 0$.*

Our proof of Theorem 1.2 fails when c is nonseparating and $g = 7$, or $g = 4k$ for $k \geq 2$ and $S \setminus c$ is orientable. We conjecture that also in these cases we have $cl_{\mathcal{T}(S)}(t_c^n) = 1$ for any $n \in \mathbb{Z}$.

2. Proofs

Consider a torus with three holes T . Let c_1, c_2, c_3 be its boundary curves and let b, a_1, a_2, a_3 be nonseparating simple closed curves in the interior of T , such that a_1, a_2, a_3 are pairwise disjoint, and b intersects a_i transversally at one point for $i = 1, 2, 3$ (Fig. 1). The right Dehn twists about these curves satisfy the following relations in the mapping class group of T :

- Twists about disjoint curves commute,
- $t_b t_{a_i} t_b = t_{a_i} t_b t_{a_i}$ for $i = 1, 2, 3$,
- $(t_b t_{a_1} t_{a_2} t_{a_3})^3 = t_{c_1} t_{c_2} t_{c_3}$.

The first two are the well known braid relations, the third is the star relation discovered by Gervais [3]. By using the braid relations we can rewrite the star relation in the following way.

$$\begin{aligned}
 t_{c_1} t_{c_2} t_{c_3} &= (t_b t_{a_1} t_{a_2} t_{a_3})(t_b t_{a_1} t_{a_2} t_{a_3})(t_b t_{a_1} t_{a_2} t_{a_3}) = t_b t_{a_2} t_{a_3} (t_{a_1} t_b t_{a_1}) t_{a_2} (t_{a_3} t_b t_{a_3}) t_{a_1} t_{a_2} \\
 &= t_b t_{a_2} t_{a_3} t_b t_{a_1} (t_b t_{a_2} t_b) t_{a_3} t_b t_{a_1} t_{a_2} = (t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2})(t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2}).
 \end{aligned}$$

Since t_{c_i} commute with all twists, for every $n \in \mathbb{Z}$ we have

$$t_{c_1}^n = (t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2} t_{c_2}^{-1})^n (t_{c_3}^{-1} t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2})^n. \tag{1}$$

There is a reflectional symmetry $r : T \rightarrow T$ such that $r(b) = b$, $r(a_1) = a_1$, $r(a_2) = a_3$, $r(c_1) = c_1$, $r(c_2) = c_3$. Since r is orientation reversing it conjugates right twists to left twists, and so we have

$$r(t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2} t_{c_2}^{-1})^{-n} r = (t_{c_3}^{-1} t_{a_3} t_{a_1} t_b t_{a_2} t_{a_3} t_b)^n.$$

By the braid relations we have

$$\begin{aligned} t_{c_3}^{-1} t_{a_3} t_{a_1} t_b t_{a_2} t_{a_3} t_b &= t_{c_3}^{-1} t_{a_1} (t_{a_3} t_b t_{a_3}) t_{a_2} t_b = t_{c_3}^{-1} t_{a_1} t_b t_{a_3} (t_b t_{a_2} t_b) \\ &= t_{c_3}^{-1} t_{a_1} t_b t_{a_3} t_{a_2} t_b t_{a_2} = t_{a_1} (t_{c_3}^{-1} t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2}) t_{a_1}^{-1}. \end{aligned}$$

Thus

$$(t_{c_3}^{-1} t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2})^n = t_{a_1}^{-1} r (t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2} t_{c_2}^{-1})^{-n} r t_{a_1},$$

and by (1)

$$t_{c_1}^n = [(t_b t_{a_2} t_{a_3} t_b t_{a_1} t_{a_2} t_{c_2}^{-1})^n, t_{a_1}^{-1} r]. \tag{2}$$

Proof of Theorem 1.1. Let S be a closed orientable surface of genus $g \geq 3$ or a closed nonorientable surface of genus $g \geq 7$. Consider the torus T as embedded in S in such a way that the reflectional symmetry r extends to $r : S \rightarrow S$. Then (2) holds in $\mathcal{M}^\circ(S)$. The embedding of T in S can be arranged in such a way that c_1 is nonseparating in S or separating, bounding a subsurface of arbitrary topological type. Moreover, if S is nonorientable of even genus and c_1 is nonseparating then $S \setminus c_1$ may be orientable or not (the former case is shown in Fig. 1). It follows that for every simple closed curve c on S there is a homeomorphism $h : S \rightarrow S$ such that $h(c) = c_1$, for appropriate embedding of T in S . Thus $t_c^n = h^{-1} t_{c_1}^n h$ for $n \in \mathbb{Z}$ and, since a conjugate of a commutator is also a commutator, we have proved Theorem 1.1. \square

Recall that for a closed nonorientable surface S of genus g , $H_1(S; \mathbb{R})$ is a real vector space of dimension $g - 1$. For $f \in \mathcal{M}(S)$ let $f_* : H_1(S; \mathbb{R}) \rightarrow H_1(S; \mathbb{R})$ be the induced automorphism. It turns out that the determinant homomorphism $f \mapsto \det f_*$ takes values in the group $\{-1, 1\}$ and its kernel is the twist subgroup $\mathcal{T}(S)$ (cf. [8] and [11, Corollary 6.3]).

Proof of Theorem 1.2. The idea of the proof is the same as for Theorem 1.1. The only problem is that the involution r may not be an element of $\mathcal{T}(S)$, in which case it has to be replaced by a different mapping class.

Suppose that S and c satisfy one of the assumptions of the theorem. Consider T as embedded in S in such a way that $c_1 = c$, as in the proof of Theorem 1.1. If c is separating, then we may arrange that the component of $S \setminus c$ which does not contain T is nonorientable of genus at least 2. If c is separating or nonseparating with $S \setminus c$ nonorientable, then $N = S \setminus T$ is a nonorientable surface of genus at least 2 and hence it supports a homeomorphism h which is not a product of Dehn twists (we may take h to be a crosscap slide or Y-homeomorphism introduced by Lickorish [8]). Since h is equal to the identity on T , thus it commutes with twists about b, a_i, c_i for $i = 1, 2, 3$. Now if the involution r , which is an extension of the reflectional symmetry of T , is in $\mathcal{T}(S)$, then $cl_{\mathcal{T}(S)}(t_c^n) = 1$ by (2). If $r \notin \mathcal{T}(S)$ then $rh \in \mathcal{T}(S)$, and $cl_{\mathcal{T}(S)}(t_c^n) = 1$ by (2) with rh in the place of r .

Now suppose that $S \setminus c_1$ is orientable. Fig. 1 shows S as being obtained from an orientable surface S' by identifying two boundary components. The homology classes of the curves $a_1, b, c_2, d, h, e_i, f_i$ for $1 \leq i \leq k$, where $g = 2(k + 3)$, form a basis of $H_1(S; \mathbb{R})$ (note that $[c_1] = 0$ in $H_1(S; \mathbb{R})$). Now we may take r as being induced by a reflection of S' , so that $r_*[a_1] = [a_1]$, $r_*[b] = -[b]$, $r_*[c_2] = [c_2]$, $r_*[d] = -[d]$, $r_*[h] = [h] - [d]$, $r_*[e_i] = -[e_i]$, $r_*[f_i] = [f_i]$ for $1 \leq i \leq k$. We see that $\det r_* = (-1)^k$, which means that $r \in \mathcal{T}(S)$ if and only if $g \equiv 2 \pmod 4$. \square

Remark. Let c be any two-sided simple closed curve on a surface S . There is a homeomorphism $h : S \rightarrow S$ preserving c and reversing orientation of its neighborhood. We have $t_c = h t_c^{-1} h^{-1}$ and

$$t_c^{2n} = t_c^n t_c^n = t_c^n h t_c^{-n} h^{-1} = [t_c^n, h]$$

for any $n \in \mathbb{Z}$. Thus any even power of any Dehn twist on any surface S is equal to a single commutator of elements of $\mathcal{M}^\circ(S)$. Moreover, if $S \setminus c$ is nonorientable of genus at least 2, then we may take $h \in \mathcal{T}(S)$ by composing it if necessary with a homeomorphism fixing c which is not a product of Dehn twists. In particular, if c is a nonseparating two-sided curve on a closed nonorientable surface of genus 7 then $scl_{\mathcal{T}(S)}(t_c) = 0$, which slightly improves Corollary 1.3.

References

[1] V. Braungardt, D. Kotschick, Clustering of critical points in Lefschetz fibrations and the symplectic Szpiro inequality, *Trans. Amer. Math. Soc.* 355 (2003) 3217–3226.
 [2] H. Endo, D. Kotschick, Bounded cohomology and non-uniform perfection of mapping class groups, *Invent. Math.* 144 (2001) 169–175.
 [3] S. Gervais, Presentation and central extensions of mapping class groups, *Trans. Amer. Math. Soc.* 348 (1996) 3097–3132.
 [4] M. Korkmaz, First homology group of mapping class group of nonorientable surfaces, *Math. Proc. Cambridge Philos. Soc.* 123 (1998) 487–499.
 [5] M. Korkmaz, B. Ozbagci, Minimal number of singular fibers in a Lefschetz fibration, *Proc. Amer. Math. Soc.* 129 (2001) 1545–1549.

- [6] M. Korkmaz, Stable commutator length of a Dehn twist, *Michigan Math. J.* 52 (2004) 23–31.
- [7] D. Kotschick, Quasi-homomorphisms and stable lengths in mapping class groups, *Proc. Amer. Math. Soc.* 132 (2004) 3167–3175.
- [8] W.B.R. Lickorish, Homeomorphisms of non-orientable two-manifolds, *Proc. Cambridge Philos. Soc.* 59 (1963) 307–317.
- [9] W.B.R. Lickorish, On the homeomorphisms of a non-orientable surface, *Proc. Cambridge Philos. Soc.* 61 (1965) 61–64.
- [10] J. Powell, Two theorems on the mapping class group of a surface, *Proc. Amer. Math. Soc.* 68 (1978) 347–350.
- [11] M. Stukow, The twist subgroup of the mapping class group a nonorientable surface, *Osaka J. Math.* 46 (2009) 717–738.