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Partial Differential Equations

Reproductive solution of a second-grade fluid system ☆

Solution reproductive d'un système de fluide de grade deux

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ABSTRACT

We treat the existence and uniqueness of reproductive solution (weak time-periodic solution) of a second-grade fluid system for small enough source terms, by using the Galerkin approximation method and compactness arguments.

RÉSUMÉ

On traite l'existence et l'unicité de la solution reproductive d'un système de fluide de grade deux avec des termes sources suffisamments petits, en utilisant la méthode d'approximation de Galerkin et des arguments de compacité.

Version française abrégée

On considère l'existence et l'unicité de la solution reproductive d'un système de fluide de grade deux (3) avec des termes sources suffisamments petits, en utilisant la méthode d'approximations de Galerkin et des arguments de compacité. En fait, d'abord nous montrons qu'il existe une suite $(\mathbf{u}_m(t))$ des approximations de Galerkin qui vérifient $\mathbf{u}_m(0) = \mathbf{u}_m(T)$. Ensuite, par des arguments de compacité, nous démontrons que la suite d'approximations de Galerkin converge vers une solution reproductive du système (3). Finalement, l'unicité de la solution (petite) est démontrée. Nous donnons le résultat principal de cette Note :

Théorème 0.1. *Soit $T > 0$ et $\Omega \subseteq \mathbb{R}^3$ un ouvert borné de classe $C^{3,1}$. Si $\|\mathbf{f}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$ et $\|\operatorname{curl}\mathbf{f}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$ sont suffisamments petits, il existe une unique solution reproductive $\mathbf{u} \in L^\infty(0, T; \mathbf{H}^3(\Omega))$ du système (3).*

1. Introduction

We consider an incompressible non-Newtonian flow of grade two in a bounded three-dimensional domain Ω and a fixed time interval $(0, T)$. From a mathematical point of view, if $\mathbf{u} : \Omega \times]0, T[\to \mathbb{R}^3$ and $p : \Omega \times]0, T[\to \mathbb{R}$ are the velocity and pressure respectively, the initial–boundary problem for an incompressible fluid of grade two is given by

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$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u})-\nu\Delta\mathbf{u}+\operatorname{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u}+\nabla q=\mathbf{f} \quad \text{in } \Omega\times\,]0,T[, \\ \operatorname{div}\mathbf{u}=0 \quad \text{in } \Omega\times\,]0,T[, \qquad \mathbf{u}=\mathbf{0} \quad \text{on } \partial\Omega\times\,]0,T[, \qquad \mathbf{u}(0)=\mathbf{u}_0 \quad \text{in } \Omega. \end{cases} \tag{1}$$

Here, $\nu>0$ represents the kinematic viscosity and $\mathbf{f}$ the external forces and q is a potential function given by $q=p-\alpha(\mathbf{u}\cdot\Delta\mathbf{u}+\frac{1}{4}|D\mathbf{u}|^2)-\frac{1}{2}\mathbf{u}\cdot\mathbf{u}$, where $(D\mathbf{u})_{ij}=\partial_j\mathbf{u}_i+\partial_i\mathbf{u}_j$ is the linear strain tensor. Moreover, $\alpha>0$ is a constant related to the non-Newtonian behavior of the fluid. Note that (1) generalizes the Navier–Stokes equations since it reduces to them when $\alpha=0$.

The study of this kind of fluids was initiated by Dunn and Fosdick in [3] and by Fosdick and Rajapogal in [4]. The first successful mathematical analysis of (1) was done by Cioranescu and Ouazar in [2]. Another work is due to Galdi and Sequeira [5], where the authors obtain some existence results.

Later, Cioranescu and Girault in [1] established existence, uniqueness and regularity of a global weak solution of (1) under hypotheses either of small data $\mathbf{f}$ and $\mathbf{u}_0$ or on small enough time intervals for arbitrary data. The existence is obtained by applying Galerkin's method associated to an adequate spectral basis, such that the Galerkin solution also satisfies the variational formulation corresponding to the following vorticity equation for $\mathbf{w}=\operatorname{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})$ (see (7) below):

$$\frac{\partial}{\partial t}\mathbf{w}-\nu\Delta\operatorname{curl}\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{w}-(\mathbf{w}\cdot\nabla)\mathbf{u}=\operatorname{curl}\mathbf{f}. \tag{2}$$

Following the ideas given in [1] and [2], we will prove the existence of reproductive solutions in $(0,T)$ for a Second-Grade fluid system, when $\mathbf{f}$ is small enough. More precisely we seek solutions of the system:

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{u}-\alpha\Delta\mathbf{u})-\nu\Delta\mathbf{u}+\operatorname{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u}+\nabla q=\mathbf{f} \quad \text{in } \Omega\times\,]0,T[, \\ \operatorname{div}\mathbf{u}=0 \quad \text{in } \Omega\times\,]0,T[, \qquad \mathbf{u}=\mathbf{0} \quad \text{on } \partial\Omega\times\,]0,T[, \qquad \mathbf{u}(0)=\mathbf{u}(T) \quad \text{in } \Omega, \end{cases} \tag{3}$$

where the usual initial condition $\mathbf{u}(0)=\mathbf{u}_0$ has been changed by the time-periodic condition $\mathbf{u}(0)=\mathbf{u}(T)$. Here, we assume that $\mathbf{f}$ depends on the time t (notice that if $\mathbf{f}$ does not depend on t, any steady-state solution of the Second-Grade fluid is actually a reproductive solution).

2. Preliminaries

In this section, we introduce the adequate notations and spaces in order to solve system (3). For more details, we refer to [6] and [7]. Let Ω be a simply-connected bounded domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathcal{C}^{3,1}$. In what follows, spaces in bold face represent spaces of three-dimensional vector functions. We define the Hilbert spaces $\mathbf{H}$ and $\mathbf{V}$ in the following manner:

$$\mathbf{H}=\{\mathbf{v}\in\mathbf{L^2}(\Omega):\ \operatorname{div}\mathbf{v}=\mathbf{0},\ \mathbf{v}\cdot\mathbf{n}=\mathbf{0}\text{ on }\partial\Omega\}, \qquad \mathbf{V}=\{\mathbf{v}\in\mathbf{H^1}(\Omega):\ \operatorname{div}\mathbf{v}=\mathbf{0},\ \mathbf{v}=\mathbf{0},\ \text{on }\partial\Omega\}.$$

For a fixed $\alpha\in\mathbb{R}^+$, we introduce the space $\mathbf{V}_2=\{\mathbf{v}\in\mathbf{V}:\ \operatorname{curl}(\mathbf{v}-\alpha\Delta\mathbf{v})\in\mathbf{L}^2(\Omega)\}$ equipped with the scalar product

$$(\mathbf{u},\mathbf{v})_{V_2}=(\mathbf{u},\mathbf{v})+\alpha(\nabla\mathbf{u},\nabla\mathbf{v})+\big(\operatorname{curl}(\mathbf{u}-\alpha\Delta\mathbf{u}),\operatorname{curl}(\mathbf{v}-\alpha\Delta\mathbf{v})\big).$$

Hereafter $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)$. In the next lemma it is proved that the semi-norm $\|\operatorname{curl}(\mathbf{v}-\alpha\Delta\mathbf{v})\|_{\mathbf{L}^2(\Omega)}$ is in fact a norm in $\mathbf{V}_2$ equivalent to the $\mathbf{H}^3$-norm.

Lemma 2.1. *(See [1, p. 320].) Let Ω be a bounded, simply-connected open set of $\mathbb{R}^3$ with a boundary of class $\mathcal{C}^{2,1}$. Then any $\mathbf{v}\in\mathbf{V}_2$ belongs to $\mathbf{H}^3(\Omega)$ and there exists a constant $C(\alpha)$ such that*

$$\|\mathbf{v}\|_{\mathbf{H}^3(\Omega)}\leqslant C(\alpha)\big\|\operatorname{curl}(\mathbf{v}-\alpha\Delta\mathbf{v})\big\|_{\mathbf{L}^2(\Omega)}, \quad \forall\mathbf{v}\in\mathbf{V}_2.$$

An easy but tedious computation gives us the following equality:

$$\big(\operatorname{curl}(\mathbf{u}-\alpha\Delta\mathbf{u})\times\mathbf{u},\mathbf{v}\big)=b(\mathbf{u};\mathbf{u},\mathbf{v})-\alpha\,b(\mathbf{u};\Delta\mathbf{u},\mathbf{v})+\alpha\,b(\mathbf{v};\Delta\mathbf{u},\mathbf{u})$$

where $b(\mathbf{u};\mathbf{v},\mathbf{w})=((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})$. From this, the variational formulation of the problem (3) is the following:

Given $\mathbf{f}\in L^\infty(0,T;\mathbf{L}^2(\Omega))$ with $\operatorname{curl}\mathbf{f}\in L^\infty(0,T;\mathbf{L}^2(\Omega))$, find $\mathbf{u}\in L^\infty(0,T;\mathbf{V}_2)$ with $\mathbf{u}'\in L^\infty(0,T;\mathbf{V})$ ($\mathbf{u}'$ denotes the time derivative of $\mathbf{u}$) such that

$$\big(\mathbf{u}',\mathbf{v}\big)+\alpha\,\big(\nabla\mathbf{u}',\nabla\mathbf{v}\big)+\nu(\nabla\mathbf{u},\nabla\mathbf{v})+b(\mathbf{u};\mathbf{u},\mathbf{v})-\alpha\,b(\mathbf{u};\Delta\mathbf{u},\mathbf{v})+\alpha b(\mathbf{v};\Delta\mathbf{u},\mathbf{u})=(\mathbf{f},\mathbf{v}), \quad \forall\mathbf{v}\in\mathbf{V}, \tag{4}$$

jointly with the time-periodic condition $\mathbf{u}(0)=\mathbf{u}(T)$.

3. Reproductive solution

By following the ideas given in [1,2] we consider the basis $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ furnished by the eigenfunctions of the problem:

$$\mathbf{v}_j \in \mathbf{V}_2, \quad (\mathbf{v}_j, \mathbf{v})_{V_2} = \lambda_j\big\{(\mathbf{v}_j, \mathbf{v}) + \alpha(\nabla \mathbf{v}_j, \nabla \mathbf{v})\big\}, \quad \forall \mathbf{v} \in \mathbf{V}_2,\ \forall j \in \mathbb{N}. \tag{5}$$

Lemma 3.1. *(See [1, p. 326].) Let Ω be a bounded simply-connected open set of $\mathbb{R}^3$ with a boundary of class $\mathcal{C}^{3,1}$. Then the eigenfunctions of problem* (5) *belong to $\mathbf{H}^4(\Omega)$.*

For every $m \in \mathbb{N}$, we define $\mathbf{V}_m$ the vector space spanned by the first m eigenfunctions $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and by P_m the orthogonal projection on $\mathbf{V}_m$ with respect to the scalar product in $\mathbf{V}_2$. In order to construct a periodic solution of the problem (3), we start considering Galerkin approximations with generic initial data, afterwards we will find reproductive Galerkin solutions and finally the existence of reproductive solutions of (3) is proved by a limit process.

Indeed, for $j \in \{1, 2, \ldots, m\}$ we consider $\mathbf{u}_m(t) = \sum_{j=1}^m c_j^m(t)\mathbf{v}_j$ solution of the initial-valued problem

$$\begin{cases} \big(\mathbf{u}_m'(t), \mathbf{v}_j\big) + \alpha\big(\nabla \mathbf{u}_m'(t), \nabla \mathbf{v}_j\big) + \nu\big(\nabla \mathbf{u}_m(t), \nabla \mathbf{v}_j\big) + b\big(\mathbf{u}_m(t); \mathbf{u}_m(t), \mathbf{v}_j\big), \\ -\alpha b\big(\mathbf{u}_m(t); \Delta \mathbf{u}_m(t), \mathbf{v}_j\big) + \alpha b\big(\mathbf{v}_j, \Delta \mathbf{u}_m(t), \mathbf{u}_m(t)\big) = \big(\mathbf{f}(t), \mathbf{v}_j\big), \quad \forall j = 1, \ldots, m, \\ \mathbf{u}_m(0) \text{ is given in } \mathbb{R}^m. \end{cases} \tag{6}$$

Setting $\mathbf{w}_m(t) = \operatorname{curl}(\mathbf{u}_m(t) - \alpha\Delta\mathbf{u}_m(t))$ and $\mathbf{z}_j = \operatorname{curl}(\mathbf{v}_j - \alpha\Delta\mathbf{v}_j)$, and multiplying (6) by λ_j, this becomes

$$\big(\mathbf{w}_m', \mathbf{z}_j\big) - \nu(\Delta \operatorname{curl} \mathbf{u}_m, \mathbf{z}_j) + b(\mathbf{u}_m; \mathbf{w}_m, \mathbf{z}_j) - b(\mathbf{w}_m; \mathbf{u}_m, \mathbf{z}_j) = (\operatorname{curl} \mathbf{f}, \mathbf{z}_j), \quad \forall j = 1, \ldots, m. \tag{7}$$

Note that the fact that $\mathbf{u}_m(t) \in \mathbf{H}^4(\Omega)$ implies that all terms in (7) belong to $L^2(\Omega)$.

The existence of these Galerkin approximations is proved in [1]. For simplicity, we denote

$$\varphi(t) = \big\|\mathbf{u}_m(t)\big\|^2_{\mathbf{L}^2(\Omega)} + \alpha\big\|\nabla \mathbf{u}_m(t)\big\|^2_{\mathbf{L}^2} \quad \text{and} \quad \psi(t) = \big\|\operatorname{curl}\big(\mathbf{u}_m(t) - \alpha\Delta\mathbf{u}_m(t)\big)\big\|^2_{L^2(\Omega)}.$$

The following result gives an estimate in the weak norm $\varphi(t)$, based on weak formulation (6):

Lemma 3.2. *(See [1, p. 327].) Solutions $\mathbf{u}_m$ of the problem* (6) *satisfy the following inequality*:

$$\varphi(t) \leqslant e^{-\nu K t}\varphi(0) + \frac{\mathcal{P}^2}{\nu}\int_0^t e^{-\nu K(t-s)}\big\|\mathbf{f}(s)\big\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}s, \quad \forall t \in [0, T], \tag{8}$$

where $\mathcal{P} > 0$ is the Poincaré constant ($\|u\|_{L^2} \leqslant \mathcal{P}\|\nabla u\|_{L^2}$ for all $u \in H_0^1(\Omega)$) and $K = (\mathcal{P}^2 + \alpha)^{-1}$.

Now, we will give some sufficient conditions to find Galerkin solutions of (6) defined in invariant sets with respect to the initial and final time data.

Theorem 3.3. *If $\varphi(0) \leqslant M_0(\mathbf{f})$, then $\varphi(t) \leqslant M_0(\mathbf{f})$, for each $t \in [0, T]$, where*

$$M_0(\mathbf{f}) = \frac{\mathcal{P}^2}{\nu^2 K}\|\mathbf{f}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))}.$$

Proof. From (8), we have that

$$\begin{aligned} \varphi(t) &\leqslant e^{-\nu K t}\varphi(0) + \big(1 - e^{-\nu K t}\big)\frac{\mathcal{P}^2}{\nu^2 K}\|\mathbf{f}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \\ &\leqslant e^{-\nu K t}M_0(\mathbf{f}) + \big(1 - e^{-\nu K t}\big)M_0(\mathbf{f}) = M_0(\mathbf{f}). \qquad \square \end{aligned}$$

The estimate given in Theorem 3.3 leads to define a continuous operator from $\mathbb{R}^m$ to $\mathbb{R}^m$, mapping the initial value $\mathbf{u}_m(0)$ to the final value $\mathbf{u}_m(T)$, giving the existence of a bounded convex invariant set to this operator. Then, one has the existence of periodic Galerkin solutions (as fixed-points of this operator). But these periodic Galerkin solutions are bounded only in $L^\infty(0, T; \mathbf{H}^1(\Omega))$, and this bound is not sufficient to control the passage to the limit (as $m \uparrow \infty$).

The following result gives an estimate in the regular norm $\psi(t)$, based on vorticity formulation (7):

Lemma 3.4. *(See [1, p. 329].) If $\partial\Omega$ is of class $C^{3,1}$, then there exists a constant $C_2(\alpha) > 0$ such that $\psi(t)$ satisfies the differential inequality in* $[0, T]$:

$$\psi'(t) + \left(\frac{\nu}{\alpha} - 2C_2(\alpha)\psi^{1/2}(t)\right)\psi(t) \leqslant \frac{4\nu}{\alpha^2} e^{-\nu K t}\varphi(t) + \frac{2\alpha}{\nu}\left\|\operatorname{curl}\mathbf{f}(t)\right\|^2_{\mathbf{L}^2(\Omega)}. \tag{9}$$

The choice of the spectral basis $(\mathbf{v}_j)_j \subset \mathbf{H}^4(\Omega)$ defined in (5) is very important in order to assure inequalities (8) and (9) (see [1] for details).

Now, we are going to deduce an invariant bound for $\psi(t)$ imposing $\mathbf{f}$ small enough.

Theorem 3.5. *Let $M_1 > 0$ be such that*

$$2C_2(\alpha)s^{1/2} < \frac{\nu}{2\alpha}, \quad \forall s \in (0, M_1] \quad \left(\textit{i.e. } M_1 < \left(\frac{\nu}{4\alpha\, C_2(\alpha)}\right)^2\right). \tag{10}$$

Furthermore, let us suppose that $\mathbf{f}$ *satisfies the smallness hypotheses*:

$$\frac{4\nu}{\alpha^2}M_0(\mathbf{f}) + \frac{2\alpha}{\nu}\|\operatorname{curl}\mathbf{f}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leqslant \frac{\nu}{2\alpha}M_1. \tag{11}$$

If $\varphi(0) \leqslant M_0(\mathbf{f})$ and $\psi(0) \leqslant M_1$, then $\psi(t) \leqslant M_1$ for all $t \in [0, T]$.

Proof. From Theorem 3.3 and hypothesis $\varphi(0) \leqslant M_0(\mathbf{f})$ one has $\varphi(t) \leqslant M_0(\mathbf{f})$ for any $t \in [0, T]$. Then, by using hypothesis (11), differential inequality (9) reduces to

$$\psi'(t) + \left(\frac{\nu}{\alpha} - 2C_2(\alpha)\psi^{1/2}(t)\right)\psi(t) \leqslant \frac{\nu}{2\alpha}M_1. \tag{12}$$

Taking into account (10), there exists $\delta > 0$ such that

$$2C_2(\alpha)s^{1/2} \leqslant \frac{\nu}{2\alpha}, \quad \forall s \in [M_1, M_1 + \delta]. \tag{13}$$

Firstly, we are going to prove the following estimate:

$$\psi(t) < M_1 + \delta, \quad \forall t \in [0, T].$$

By contradiction, let $T^* \in (0, T]$ be the first value such that $\psi(T^*) = M_1 + \delta$ and $\psi(t) < M_1 + \delta$, $\forall t \in [0, T^*[$. In particular, from (10) and (13), $2C_2(\alpha)\psi(t)^{1/2} \leqslant \nu/(2\alpha)$ for all $t \in [0, T^*]$. Thus, from (12)

$$\psi'(t) + \frac{\nu}{2\alpha}\psi(t) \leqslant \frac{\nu}{2\alpha}M_1, \quad \forall t \in \left(0, T^*\right]. \tag{14}$$

By multiplying the above inequality by $e^{\frac{\nu}{2\alpha}t}$, integrating in time for $t \in [0, T^*]$ and using hypothesis $\psi(0) \leqslant M_1$, we have that $\psi(T^*) \leqslant e^{-\frac{\nu}{2\alpha}T^*}M_1 + (1 - e^{-\frac{\nu}{2\alpha}T^*})M_1 = M_1$, which is a contradiction, therefore $\psi(t) < M_1 + \delta$ for all $t \in [0, T]$.

In particular, differential inequality (14) holds for any $t \in [0, T]$, hence by repeating the same arguments in each interval $[0, t]$ for all $t \in [0, T]$, we get $\psi(t) \leqslant e^{-\frac{\nu t}{2\alpha}}M_1 + (1 - e^{-\frac{\nu t}{2\alpha}})M_1 = M_1$, which finish the proof. □

Now, we are in position to prove the main result of existence and uniqueness of reproductive solution (Theorem 0.1).

With respect to the existence, for every $(\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m$ and $\mathbf{u} = \xi_1\mathbf{v}_1 + \xi_2\mathbf{v}_2 + \cdots + \xi_m\mathbf{v}_m \in \mathbf{V}_m$, we define the following equivalent norms:

$$\left\|(\xi_1, \xi_2, \ldots, \xi_m)\right\|_{a,\mathbb{R}^m} := \left(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \alpha\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\right)^{1/2},$$
$$\left\|(\xi_1, \xi_2, \ldots, \xi_m)\right\|_{b,\mathbb{R}^m} := \left\|\operatorname{curl}(\mathbf{u} - \alpha\Delta\mathbf{u})\right\|_{\mathbf{L}^2(\Omega)}.$$

Given $(\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m$, we define the operator $\Phi^m : \mathbb{R}^m \to \mathbb{R}^m$ in the following manner: $\Phi^m(\xi_1, \xi_2, \ldots, \xi_m) = (c_1^m(T), c_2^m(T), \ldots, c_m^m(T)) \in \mathbb{R}^m$, where $(c_j^m(t))_m$ are the coefficients of the expansion in $\mathbf{V}_m$ of $\mathbf{u}_m(t)$ the solution of (6) with the initial condition $\mathbf{u}_m(0) = \xi_1\mathbf{v}_1 + \xi_2\mathbf{v}_2 + \cdots + \xi_m\mathbf{v}_m$. Note that Φ^m is a continuous operator because problem (6) can be reformulated as a Cauchy problem related to an ordinary differential system written in normal form, i.e.

$$\frac{\mathrm{d}\Phi^m(t)}{\mathrm{d}t} = g\left(\Phi^m(t)\right), \quad t \in (0, T), \qquad \Phi^m(0) = (\xi_1, \xi_2, \ldots, \xi_m),$$

and this Cauchy problem is continuous with respect to the initial condition $(\xi_1, \xi_2, \ldots, \xi_m)$.

Let the following compact and convex set of $\mathbb{R}^m$:

$$\bar{B} = \left\{(\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m:\ \left\|(\xi_1, \xi_2, \ldots, \xi_m)\right\|_{a,\mathbb{R}^m} \leqslant M_0 \text{ and } \left\|(\xi_1, \xi_2, \ldots, \xi_m)\right\|_{b,\mathbb{R}^m} \leqslant M_1\right\}$$

where M_0 and M_1 are the constants given in Theorems 3.3 and 3.5 respectively. From Theorems 3.3 and 3.5, we have that Φ^m maps $\bar{B}$ into $\bar{B}$. From Brouwer fixed-point theorem, we deduce that there exists a fixed point of Φ^m and consequently, there exists a reproductive Galerkin solution $\mathbf{u}_m$ (such that $\mathbf{u}_m(0) = \mathbf{u}_m(T)$).

From Theorem 3.5, the sequence of reproductive Galerkin solutions $(\mathbf{u}_m)_{m\geqslant 1}$ is bounded in $L^\infty(0, T; \mathbf{V}_2)$. By adapting the proof of Lemma 4.5 in [1], we obtain the following lemma:

Lemma 3.6. *Let $\mathbf{f} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$ with $\operatorname{curl}\mathbf{f} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$ and assume that the sequence $(\mathbf{u}_m)_{m\geqslant 1}$ is bounded (with respect to m) in $L^\infty(0, T; \mathbf{H}^3(\Omega))$. Then $(\mathbf{u}'_m)_{m\geqslant 1}$ is bounded in $L^\infty(0, T; \mathbf{H}^1(\Omega))$.*

Then, by applying a compactness theorem (of Aubin–Lions' type), there exists a subsequence of $(\mathbf{u}_m)_{m\geqslant 1}$ that converges to $\mathbf{u}$ a solution of (3), in the weak sense of (4), hence the existence of reproductive solution is proved.

With respect to the uniqueness, assuming small enough data $\mathbf{f}$ and an additional condition on M_1, we will have that this reproductive solution $\mathbf{u}$ of (3) obtained by the Galerkin procedure is unique. Indeed, let $\tilde{\mathbf{u}} \in L^\infty(0, T; \mathbf{V}_2)$ be a reproductive solution of problem (3). Then, the function difference $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$ satisfies (see [1, p. 326])

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\mathbf{v}(t)\right\|^2_{L^2(\Omega)} + \alpha\left|\mathbf{v}(t)\right|^2_{H^1(\Omega)}\right) + \nu\left|\mathbf{v}(t)\right|^2_{H^1(\Omega)} + b(\mathbf{v}; \mathbf{u}, \mathbf{v}) + \alpha b(\mathbf{v}; \Delta\mathbf{v}, \mathbf{u}) - \alpha b(\mathbf{u}; \Delta\mathbf{v}, \mathbf{v}) \\ &\quad\leqslant \left|\mathbf{u}(t)\right|_{H^1(\Omega)}\left\|\mathbf{v}(t)\right\|^2_{L^4(\Omega)} + 2\alpha\left\|\nabla\mathbf{u}(t)\right\|_{L^\infty(\Omega)}\left|\mathbf{v}(t)\right|^2_{H^1(\Omega)} + \alpha\left\|\partial^2\mathbf{u}(t)\right\|_{L^4(\Omega)}\left\|\mathbf{v}(t)\right\|_{L^4(\Omega)}\left|\mathbf{v}(t)\right|_{H^1(\Omega)}. \end{aligned} \tag{15}$$

Then, by using the estimates already obtained for $\mathbf{u}$:

$$\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \leqslant M_0/\alpha, \qquad \|\mathbf{u}\|_{L^\infty(0,T;\mathbf{H}^3(\Omega))} \leqslant C(\alpha)M_1,$$

where $C(\alpha)$ is the constant given in Lemma 2.1, the left-hand side of (15) can be bounded by

$$\left(C_1^2\frac{M_0(\mathbf{f})}{\alpha} + 2\alpha C_2 C(\alpha)M_1 + \alpha C_2 C_1^2 C(\alpha)M_1\right)\left|\mathbf{v}(t)\right|^2_{H^1(\Omega)}$$

where C_1 and C_4 are Sobolev imbedding constants. By choosing $\|\mathbf{f}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$ and $\|\operatorname{curl}\mathbf{f}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$ small such that M_0 and M_1 are small enough verifying $C_1^2\frac{M_0(\mathbf{f})}{\alpha} + 2\alpha C_2 C(\alpha)M_1 + \alpha C_2 C_1^2 C(\alpha)M_1 \leqslant \frac{\nu}{2}$, from (15) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\mathbf{v}(t)\right\|^2_{L^2(\Omega)} + \alpha\left|\mathbf{v}(t)\right|^2_{H^1(\Omega)}\right) + \nu\left|\mathbf{v}(t)\right|^2_{H^1(\Omega)} \leqslant 0$$

hence, integrating in $(0, T)$ and applying the time-periodicity, the uniqueness is proved.

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