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Algebraic Geometry

Einstein–Hermitian connection on twisted Higgs bundles

Connexions d'Einstein–Hermite sur les fibrés de Higgs tordus

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ABSTRACT

Let X be a smooth projective variety over $\mathbb{C}$. We prove that a twisted Higgs vector bundle $(\mathcal{E}, \theta)$ on X admits an Einstein–Hermitian connection if and only if $(\mathcal{E}, \theta)$ is polystable. A similar result for twisted vector bundles (no Higgs fields) was proved in Wang [10]. Our approach is simpler.

RÉSUMÉ

Soit X une variété projective lisse sur $\mathbb{C}$. Nous démontrons qu'un fibré de Higgs tordu $(\mathcal{E}, \theta)$ sur X possède une connexion d'Einstein–Hermite si et seulement si $(\mathcal{E}, \theta)$ est polystable. Un résultat analogue pour les fibrés vectoriels (dépourvus d'un champ de Higgs) a été démontré dans Wang [10]. Notre approche est plus simple.

1. Introduction

Donaldson, Uhlenbeck and Yau proved that a vector bundle on a complex projective manifold admits an Einstein–Hermitian connection if and only if it is polystable [3,9]. A generalization of Einstein–Hermitian connections for Higgs bundles was formulated by Hitchin (for curves) and Simpson (higher dimensions). They proved that a Higgs bundle $(\mathcal{E}, \theta)$ admits an Einstein–Hermitian connection if and only if it is polystable [4,8].

Our aim here is to establish a similar result for twisted sheaves on a smooth complex projective variety. Let X be an irreducible smooth projective variety over $\mathbb{C}$. A *twisted vector bundle* on X is a pair $(\mathcal{X}, \mathcal{E})$, where

$$\mathcal{X} \to X$$

is a gerbe banded by μ_n (the n-th roots of unity) for some n, and $\mathcal{E}$ is a vector bundle over $\mathcal{X}$; see [7,6,5,11] for twisted bundles. A twisted Higgs bundle on X is a twisted vector bundle together with a Higgs field on it.

We prove that a twisted Higgs bundle on X admits an Einstein–Hermitian connection if and only if it is polystable (see Theorem 3.1).

Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. Theorem 3.1 generalizes to twisted Higgs principal G-bundles (this is explained at the end).

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In [10], Wang proved a similar result for twisted vector bundles without Higgs structure.

2. Twisted Higgs bundles

The base field will be $\mathbb{C}$. For any positive integer n, by μ_n we will denote the finite subgroup of $\mathbb{C}^*$ consisting of the n-th roots of 1.

Let X be an irreducible smooth projective variety. Let

$$f : \mathcal{X} \to X \tag{1}$$

be a gerbe banded by μ_n. The cotangent bundle of $\mathcal{X}$ will be denoted by $\Omega^1_{\mathcal{X}}$. For any nonnegative integer i, let $\Omega^i_{\mathcal{X}} := \bigwedge^i \Omega^1_{\mathcal{X}}$ be the i-th exterior power.

Let

$$\mathcal{E} \to \mathcal{X}$$

be a vector bundle. Let $End(\mathcal{E}) := \mathcal{E} \otimes \mathcal{E}^*$ be the endomorphism bundle. The associative algebra structure of $End(\mathcal{E})$ and the exterior algebra structure of $\bigoplus_{i \geqslant 0} \Omega^i_{\mathcal{X}}$ together define an algebra structure on $End(\mathcal{E}) \otimes (\bigoplus_{i \geqslant 0} \Omega^i_{\mathcal{X}})$.

A *Higgs field* on $\mathcal{E}$ is a section θ of $End(\mathcal{E}) \otimes \Omega^1_{\mathcal{X}}$ such that the section $\theta \wedge \theta$ of $End(\mathcal{E}) \otimes \Omega^2_{\mathcal{X}}$ vanishes identically.

A *Higgs bundle* on $\mathcal{X}$ is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a vector bundle on $\mathcal{X}$, and θ is a Higgs field on $\mathcal{E}$. A Higgs bundle on $\mathcal{X}$ will be called a *twisted Higgs bundle* on X. Given a Higgs bundle $(\mathcal{E}, \theta)$ on $\mathcal{X}$, a coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ will be called a *Higgs subsheaf* if $\theta(\mathcal{F}) \subset \mathcal{F} \otimes \Omega^1_{\mathcal{X}}$.

Let G be a complex linear algebraic group. A *Higgs G-bundle* on X is a principal G-bundle $E_G \to X$ and a section $\beta \in H^0(X, \mathrm{ad}(E_G) \otimes \Omega^1_X)$ such that $\beta \wedge \beta = 0$, where $\mathrm{ad}(E_G)$ is the adjoint vector bundle.

Fix a very ample line bundle L over X. The *degree* of a torsionfree coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ will be defined to be $\mathrm{degree}((\det \mathcal{F})^{\otimes n})/n^2 \in \mathbb{Q}$. Note that $(\det \mathcal{F})^{\otimes n}$ descends to a line bundle on X; its degree is computed using L. Fix a Kähler form ω_X on X representing $c_1(L)$. Since the morphism f in (1) in étale, the pullback

$$\omega_{\mathcal{X}} := f^* \omega_X \tag{2}$$

is a Kähler form on $\mathcal{X}$. A Higgs bundle $(\mathcal{E}, \theta)$ is called *stable* (respectively, *semistable*) if for every Higgs subsheaf $\mathcal{F}$ with $1 \leqslant \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$, the inequality

$$\frac{\mathrm{degree}(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} < \frac{\mathrm{degree}(\mathcal{E})}{\mathrm{rank}(\mathcal{E})} \left(\text{respectively, } \frac{\mathrm{degree}(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} \leqslant \frac{\mathrm{degree}(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}\right)$$

holds. A semistable Higgs bundle is called *polystable* if it is a direct sum of stable Higgs bundles.

For any vector bundle $\mathcal{E} \to \mathcal{X}$, we have a decomposition $\mathcal{E} = \bigoplus_{\chi \in \mu_n^*} \mathcal{E}_\chi$. Henceforth, we will consider vector bundles $\mathcal{E}$ with $\mathcal{E}_\chi \neq 0$ for at most one character χ.

Define the homomorphism

$$\rho : \mathrm{GL}(r, \mathbb{C}) \to \mathrm{PGL}(r, \mathbb{C}) \times \mathbb{G}_m =: H \tag{3}$$

by sending A to the class of A and to $(\det A)^n$. Given a vector bundle $\mathcal{E} \to \mathcal{X}$, the extension of its structure group along ρ defines a principal H-bundle $\mathcal{E}_H \to \mathcal{X}$. Since the inertia μ_n acts trivially on $\mathcal{E}_H$, it descends to a principal H-bundle $E_H \to X$. A Higgs field θ on $\mathcal{E}$ induces a Higgs field θ_H on $\mathcal{E}_H$. This Higgs field θ_H on $\mathcal{E}_H$ descends to a Higgs field on E_H, which we again denote by θ_H.

The definitions of Higgs (semi)stable and polystable principal bundles are recalled in [1, p. 551], [2].

Lemma 2.1. *A Higgs bundle $(\mathcal{E}, \theta)$ on $\mathcal{X}$ is polystable if and only if the induced Higgs H-bundle (E_H, θ_H) on X is polystable.*

Proof. The central isogeny ρ in (3) produces a bijection of parabolic subgroups. For any parabolic subgroup $P \subset \mathrm{GL}(r, \mathbb{C})$, there is a natural bijective correspondence between the reductions of structure group of the principal $\mathrm{GL}(r, \mathbb{C})$-bundle $\mathcal{E}$ to P over any open subset $f^{-1}(U)$ and the reductions of structure group of the principal H-bundle E_H to $\rho(P)$ over U. This bijection proves the lemma. □

3. Einstein–Hermitian connection on polystable twisted Higgs bundles

A *Hermitian structure* on a vector bundle $\mathcal{E}$ on $\mathcal{X}$ is a smooth inner product on the fibers which is invariant under the action of μ_n on the fibers of $\mathcal{E}$. A Hermitian structure on $\mathcal{E}$ produces a C^∞ complex connection on $\mathcal{E}$. Let $(\mathcal{E}, \theta)$ be a Higgs bundle. An *Einstein–Hermitian connection* on $(\mathcal{E}, \theta)$ is a Hermitian structure on $\mathcal{E}$ such that corresponding connection ∇ on $\mathcal{E}$ has the following property:

$$\Lambda_{\omega_{\mathcal{X}}}\big(\mathrm{Curv}(\nabla) + [\theta, \theta^*]\big) = c \cdot \mathrm{Id}_{\mathcal{E}},$$

for some constant scalar c, where $\Lambda_{\omega_{\mathcal{X}}}$ is the adjoint of multiplication by the Kähler form $\omega_{\mathcal{X}}$ (see (2)), $\mathrm{Curv}(\nabla)$ is the curvature of ∇, and θ^* is the adjoint of θ constructed using the Hermitian form on $\mathcal{E}$.

Theorem 3.1. *Let $(\mathcal{E}, \theta)$ be a twisted Higgs bundle on X. Then $(\mathcal{E}, \theta)$ is polystable if and only if it admits an Einstein–Hermitian connection.*

Proof. Let $(\mathcal{E}, \theta)$ be a Higgs bundle on $\mathcal{X}$. First assume that $(\mathcal{E}, \theta)$ is polystable. From Lemma 2.1 we know that the induced Higgs H-bundle (E_H, θ_H) on X is polystable. A polystable Higgs H-bundle on X admits an Einstein–Hermitian connection [8,1]. Since (E_H, θ_H) is the descent of $(\mathcal{E}_H, \theta_H)$, an Einstein–Hermitian connection on (E_H, θ_H) produces an Einstein–Hermitian connection on $(\mathcal{E}_H, \theta_H)$. A connection on $\mathcal{E}_H$ defines connection on $\mathcal{E}$ because the homomorphism of Lie algebras

$$\mathrm{Lie}\big(\mathrm{GL}(r, \mathbb{C})\big) \to \mathrm{Lie}(H)$$

induced by the homomorphism ρ in (3) is an isomorphism. The connection on $(\mathcal{E}, \theta)$ induced by an Einstein–Hermitian connection on $(\mathcal{E}_H, \theta_H)$ is clearly Einstein–Hermitian.

Conversely, an Einstein–Hermitian connection on $(\mathcal{E}, \theta)$ induces an Einstein–Hermitian connection on the associated Higgs H-bundle $(\mathcal{E}_H, \theta_H)$, which, in turn, induces an Einstein–Hermitian connection on the descended Higgs H-bundle (E_H, θ_H). Therefore, the Higgs H-bundle (E_H, θ_H) is polystable. Hence from Lemma 2.1 we conclude that the Higgs bundle $(\mathcal{E}, \theta)$ is polystable. □

Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. Let Z be the center of G; define $G' := [G, G]$.

The above theorem holds for principal Higgs G-bundles on $\mathcal{X}$. The proof is the same, but, instead of the homomorphism (3), we use the homomorphism

$$\rho : G \to H := G/Z \times \big(G/G'\big) : g \mapsto \big(p(g), q(g)^n\big),$$

where $p : G \to G/Z$ and $q : G \to G/G'$ are the natural projections. Note that $G/G' \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$.

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