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Differential Geometry

Some characterizations of the Wulff shape

Sur certaines caractérisations des formes de Wulff

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ABSTRACT

For a positive function F on S^n which satisfies a suitable convexity condition, we consider the r-th anisotropic mean curvature for hypersurfaces in $\mathbb{R}^{n+1}$ which is a generalization of the usual r-th mean curvature H_r. By using an integral formula of Minkowski type for compact hypersurface due to H.J. He and H. Li, we introduce some new characterizations of the Wulff shape.

RÉSUMÉ

Étant donné une fonction positive F sur S^n qui vérifie une condition de convexité convenable, nous considérons la r-ième courbure moyenne anisotrope pour les hypersurfaces de $\mathbb{R}^{n+1}$ qui est une généralisation de la r-ième courbure moyenne usuelle H_r. En utilisant une formule intégrale de type Minkowski pour les hypersurfaces compactes due à H.J. He et H. Li, nous introduisons de nouvelles caractérisations des formes de Wulff.

1. Introduction

Let $F : S^n \to \mathbb{R}^+$ be a smooth function which satisfies the following convexity condition:

$$\left(D^2 F + F1\right)_x > 0, \quad \forall x \in S^n, \tag{1}$$

where $D^2 F$ denotes the intrinsic Hessian of F on S^n, 1 denotes the identity on $T_x S^n$, > 0 means that the matrix is positive definite.

We consider the map

$$\phi : S^n \to \mathbb{R}^{n+1},$$
$$x \mapsto F(x)x + (\operatorname{grad}_{S^n} F)_x,$$

its image $W_F = \phi(S^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of F (see [1,6,7,9]).

When M^n is compact convex hypersurface, the following characterization of the Wulff shape is recently known, also for Riemannian case (see [5]):

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Theorem 1.1. *(See [4, Theorem 1.4].) Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional compact convex hypersurface, $F : S^n \to \mathbb{R}^+$ be a smooth function which satisfies (1). If $\frac{M_r}{M_k} = const.$ for some k and r, with $0 \leqslant k < r \leqslant n$, then $X(M)$ is the Wulff shape up to translations and homotheties.*

We generalize Theorem 1.1 in the following way:

Theorem 1.2. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional compact convex hypersurface, $F : S^n \to \mathbb{R}^+$ be a smooth function which satisfies (1). If there are nonnegative constants $C_1, \dots, C_r$ such that*

$$M_r = \sum_{i=0}^{r-1} C_i M_i,$$

then $X(M)$ is the Wulff shape up to translations and homotheties.

In addition, using the integral formula of Minkowski type in [4], we have the following theorem:

Theorem 1.3. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional compact convex orientable hypersurface, $F : S^n \to \mathbb{R}^+$ be a smooth function which satisfies (1). If there is an integer r, $1 \leqslant r \leqslant n$ such that either $\langle x, \nu\rangle \leqslant -F\frac{M_{r-1}}{M_r}$ or $\langle x, \nu\rangle \geqslant -F\frac{M_{r-1}}{M_r}$ throughout M, then $X(M)$ is a Wulff shape.*

This theorem is an anisotropic version of the theorem in [2].

2. Preliminaries

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a compact, orientable hypersurface without boundary. Let $\nu : M \to S^n$ denote its Gauss map, ν is a unit inner normal vector of M.

Remember that $A_F = D^2F + F1$, $S_F = -A_F \circ \mathrm{d}\nu$. Here S_F is called the *F-Weingarten operator*, and the eigenvalues of S_F are called *anisotropic principal curvatures*. Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\lambda_1, \dots, \lambda_n$

$$\sigma_r = \sum_{i_1 <, \dots, < i_r} \lambda_{i_1} \dots \lambda_{i_r} \quad (1 \leqslant r \leqslant n).$$

We set $\sigma_0 = 1$. The r-th anisotropic mean curvature M_r is defined by

$$M_r = \frac{\sigma_r}{C_n^r}, \quad C_n^r = \frac{n!}{r!(n-r)!}$$

which was introduced by Reilly in [8].

The following Minkowski formula will be essential to proof of Theorems 1.2 and 1.3:

Lemma 2.1. *(See [4, Theorem 1.1].) Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional compact hypersurface, $F : S^n \to \mathbb{R}^+$ be a smooth function which satisfies (1). Then the following integral formulas of Minkowski type hold*:

$$\int_M \big(FM_r + M_{r+1}\langle x, \nu\rangle\big)\,\mathrm{d}A_X = 0, \quad r = 0, \dots, n-1. \tag{2}$$

The following lemmas will also be used in the sequel:

Lemma 2.2. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional compact convex hypersurface without boundary, $F : S^n \to \mathbb{R}^+$ be a smooth function which satisfies (1).*

(i) *It holds that*

$$\frac{M_i}{M_r} \geqslant \frac{M_{i-1}}{M_{r-1}}, \quad i \leqslant r. \tag{3}$$

The equality holds if and only if all the anisotropic principal curvatures are the same.

(ii) *If there are nonnegative constants $C_1, \dots, C_{r-1}$ such that $M_r = \sum_{i=1}^{r-1} C_i M_i$, then*

$$M_{r-1} \geqslant \sum_{i=1}^{r-1} C_i M_{i-1}, \tag{4}$$

and if furthermore, the equality holds then all the anisotropic principal curvatures are the same.

Proof. (i) From the convexity on M all the principal curvatures of M are positive, so all the anisotropic principal curvatures are positive, we have $M_r > 0$, $0 \leqslant r \leqslant n$ on M. By [3] we can write

$$M_{i-1}M_{i+1} \leqslant M_i^2, \dots, M_{r-2}M_r \leqslant M_{r-1}^2, \tag{5}$$

and equality holds in (5) if and only if $\lambda_1 = \cdots = \lambda_n$.

We can easily check

$$M_i M_{r-1} \geqslant M_r M_{i-1},$$

that is,

$$\frac{M_i}{M_r} \geqslant \frac{M_{i-1}}{M_{r-1}}.$$

(ii) Since $M_r = \sum_{i=1}^{r-1} C_i M_i$ and $M_r > 0$, by (3),

$$1 = \sum_{i=1}^{r-1} C_i \frac{M_i}{M_r} \geqslant \sum_{i=1}^{r-1} C_i \frac{M_{i-1}}{M_{r-1}}$$

or

$$M_{r-1} - \sum_{i=1}^{r-1} C_i M_{i-1} \geqslant 0. \tag{6}$$

If the equality holds, we have

$$\frac{M_i}{M_r} = \frac{M_{i-1}}{M_{r-1}},$$

which implies that all the anisotropic principal curvatures are the same. □

Lemma 2.3. *(See [4, Lemma 3.4].) If $\lambda_1 = \cdots = \lambda_n = const \neq 0$, then $X(M)$ is the Wulff shape up to translations and homotheties.*

3. Proofs of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. From (2), (6) and integrating

$$FM_{r-1} \geqslant \sum_{i=1}^{r-1} C_i F M_{i-1} \tag{7}$$

over M. We get

$$\begin{aligned} 0 &\leqslant \int \Big(FM_{r-1} - \sum C_i F M_{i-1}\Big)\, \mathrm{d}A_X \\ &= -\int M_r \langle x, \nu\rangle \, \mathrm{d}A_X - \sum C_i \int F M_{i-1}\, \mathrm{d}A_X \\ &= -\int M_r \langle x, \nu\rangle \, \mathrm{d}A_X + \sum C_i \int M_i \langle x, \nu\rangle \, \mathrm{d}A_X \\ &= -\int \Big(M_r - \sum C_i M_i\Big)\langle x, \nu\rangle \, \mathrm{d}A_X = 0. \end{aligned}$$

Then, by the assumptions we have

$$M_{r-1} = \sum_{i=1}^{r-1} C_i M_{i-1} \tag{8}$$

on M. Hence by Lemma 2.2 and Lemma 2.3, all the anisotropic principal curvatures are equal. That is $X(M)$ is the Wulff shape. □

Proof of Theorem 1.3. Since $M_k > 0$, the conditions $\langle x, \nu\rangle \leqslant -F\frac{M_{r-1}}{M_r}$ or $\langle x, \nu\rangle \geqslant -F\frac{M_{r-1}}{M_r}$ are respectively equivalent to $M_k\langle x, \nu\rangle + FM_{k-1} \leqslant 0$ and $M_k\langle x, \nu\rangle + FM_{k-1} \geqslant 0$. Together with either two inequalities and by (2) for $r = k-1$ we have,

$$\int \big(FM_{k-1} + M_k \langle x, \nu\rangle\big)\, \mathrm{d}A_x = 0.$$

This equality implies that $\langle x, \nu\rangle = -F\frac{M_{k-1}}{M_k}$. Substituting this value of $\langle x, \nu\rangle$ in Eq. (2) for $r = k$, we obtain

$$\int \frac{1}{M_k}\big(FM_k^2 - FM_{k+1}M_{k-1}\big)\, \mathrm{d}A_X = 0.$$

Due to the convexity of the function F and [3] we get

$$M_k^2 - M_{k+1}M_{k-1} = 0.$$

Then all the anisotropic principal curvatures are the same at all points of M. From Lemma 2.3 $X(M)$ is the Wulff shape up to translations and homotheties. □

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