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Dynamical Systems

Geometry of the common dynamics of Pisot substitutions with the same incidence matrix

Géométrie commune des substitutions Pisot de même matrice d'incidence

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A B S T R A C T

Any Pisot substitution can be associated with a bounded set with interesting properties, called the Rauzy fractal. This set is obtained by projection of the broken line associated with an infinite fixed point. Two substitutions having the same incidence matrix can have different Rauzy fractals. We show that under weak conditions, the intersection of these two fractals has strictly positive measure, and can also be generated by a substitution.

R É S U M É

À toute substitution Pisot, on sait associer, par projection de la ligne brisée associée à un point fixe infini de la substitution, un ensemble borné aux propriétés intéressantes, appelé fractal de Rauzy. Deux substitutions ayant la même matrice d'incidence peuvent avoir des fractals de Rauzy très différents. Nous montrons que, sous des conditions faibles, l'intersection de ces deux fractals est de mesure non nulle, et peut aussi être engendrée par une substitution.

1. Introduction

Let $\mathcal{A}$ be a finite set of cardinality d. A substitution σ on $\mathcal{A}$ is a morphism of the free monoid $\mathcal{A}^*$ such that the image of each letter of $\mathcal{A}$ is a nonempty word. It naturally extends to the set of one-sided sequences, denoted by $\mathcal{A}^{\mathbb{N}}$.

Let $f : \mathcal{A}^* \mapsto \mathbb{Z}^d : w \mapsto (|w|_a)_{a\in\mathcal{A}}$ be the natural homomorphism obtained by abelianization of the free monoid, called the abelianization map. We associate to every substitution σ its incidence matrix M obtained by abelianization, $M_{i,j} = |\sigma(j)|_i$. A substitution σ is primitive if there exists an integer k such that $M^k > 0$. We say that σ is an irreducible Pisot substitution if there exists one eigenvalue of M which is strictly greater than 1 and all other eigenvalues are strictly less than 1 in modulus. An equivalent definition is that the largest eigenvalue is a Pisot number, and the characteristic polynomial is irreducible. A substitution is unimodular if the determinant of its incidence matrix is equal ± 1.

Let σ be a primitive substitution, then there exists a finite number of periodic points u. We associate to u a symbolic dynamical system (Ω_u, S) where S is the shift map on $\mathcal{A}^{\mathbb{N}}$ defined by $S(a_0a_1\cdots) = a_1a_2\cdots$ and Ω_u is the closure of $\{S^m(u)\colon\ m \geqslant 0\}$ in $\mathcal{A}^{\mathbb{N}}$.

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In the case of unimodular Pisot substitutions, the symbolic dynamical system can be understood in a geometrical way. In [5], G. Rauzy proved that the dynamical system generated by the substitution $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$, is measure-theoretically conjugate to an exchange of domains in a compact set $\mathcal{R}$ of the complex plane with a self-similar structure. The construction goes as follows:

Using the abelianization map, to any finite or infinite word u, we can associate a canonical stepped line in $\mathbb{R}^d$ as a sequence $(f(P_k))$, where P_k is the prefix of length k of u. The canonical stepped line associated with a fixed point of a Pisot substitution remains within a bounded distance to the expanding direction (given by the right Perron–Frobenius eigenvector of M). Projecting the vertices of this canonical stepped line by the projection π_s on the contracting subspace E_s of the incidence matrix of σ along its expanding direction, and taking the closure of this set, yields the Rauzy fractal [2]. The projection of the stepped line can be seen as a map π from the orbit $\{S^n(u)\}$ of the fixed point to the Rauzy fractal. This map can be extended by continuity to a map $\pi : \Omega_u \to \mathcal{R}$, see [3]. In many cases, this map is known to be a measurable isomorphism.

The Rauzy fractal is a nonempty compact set which is the closure of its interior and decomposes in a natural way in d subtiles $\mathcal{R}(1), \ldots, \mathcal{R}(d)$. It is also known that these tiles induce a multiple tiling of the contracting plane. For more detail see [6, Chap. 3].

Definition 1.1. An exclusive inner point of the Rauzy fractal is a point from one subtile $\mathcal{R}_i$, $i \in \{1, \ldots, d\}$, which does not belong to any other tile from the multiple tiling. See [6, Chap. 4].

We can generalize the definition of the Rauzy fractal with the projection method:

Definition 1.2. A substitutive set is the closure of the projection of a canonical stepped line associated with a primitive substitution σ on a contracting space of the incidence matrix of σ. See [1].

To a Pisot matrix, there correspond many substitutions since there are many words with the same abelianization, see [7]. A classic example is given by the Tribonacci substitution and the flipped Tribonacci substitution, i.e.,

$$\sigma_1 : \begin{cases} a \to ab \\ b \to ac \\ c \to a \end{cases} \quad \text{and} \quad \sigma_2 : \begin{cases} a \to ab \\ b \to ca \\ c \to a \end{cases}.$$

The Rauzy fractal of the first substitution is simply connected [5], while it is a well known fact that the second fractal is not simply connected [4].

In this Note we study the common dynamics of two irreducible and unimodular Pisot substitutions σ_1 and σ_2 having the same incidence matrix (see Fig. 1). We show a geometric realization of the intersection of the interior of the two Rauzy fractals associated with σ_1 and σ_2. The main result of this Note is the following:

Theorem 1.3. *Let σ_1 and σ_2 be two irreducible and unimodular Pisot substitutions with the same incidence matrix. We denote $\mathcal{R}_1$, $\mathcal{R}_2$ their two associated Rauzy fractals. We suppose that* 0 *is an exclusive inner point of $\mathcal{R}_1$. Then the closure of the intersection of the interiors of $\mathcal{R}_1$ and $\mathcal{R}_2$ has strictly positive measure and is a substitutive set.*

2. Morphism generating the common points of two Pisot substitutions

Let σ_1 and σ_2 be two unimodular and irreducible Pisot substitutions with the same incidence matrix and $\mathcal{R}_1$, $\mathcal{R}_2$ their two associated Rauzy fractals. We denote by $\mathcal{R}$ the closure of the intersection of the interiors of $\mathcal{R}_1$ and $\mathcal{R}_2$. Let (Ω_{σ_1}, S) and (Ω_{σ_2}, S) be the symbolic dynamical systems associated with σ_1 and σ_2. We consider π_1 (resp. π_2) the projection map from the symbolic dynamical system (X_{σ_1}, S) (resp. (X_{σ_2}, S)) on the corresponding Rauzy fractal.

Lemma 2.1. *Suppose that* 0 *is an inner point of $\mathcal{R}_1$. Then $\mathcal{R}$ has non-empty interior and strictly positive Lebesgue measure.*

Proof. Because 0 is an inner point of $\mathcal{R}_1$, there exists an open set U such that $0 \in U \subset \mathcal{R}_1$. The Rauzy fractal is the closure of its interior [6] and 0 is a point of $\mathcal{R}_2$, hence there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ from the interior of $\mathcal{R}_2$ which converges to 0. Then there exist open sets V_n such that $x_n \in V_n \subset \mathcal{R}_2$. Since (x_n) converges to 0, there exists $N \in \mathbb{N}$ such that $x_N \in U$.

The open set $U \cap V_N$ is non-empty and $U \cap V_N \subset \mathcal{R}_1 \cap \mathcal{R}_2$. This implies that $\mathcal{R}$ contains a non-empty open set, hence it has strictly positive Lebesgue measure. □

We denote by Γ the group $\{\sum_{i=1}^d n_i e_i \colon \sum_{i=1}^d n_i = 0, n_i \in \mathbb{Z}\} \subset \mathbb{Z}^d$ where $(e_1, \ldots, e_d)$ denotes the canonical base of $\mathbb{R}^d$. We introduce the following lemma from [6, Chap. 4].

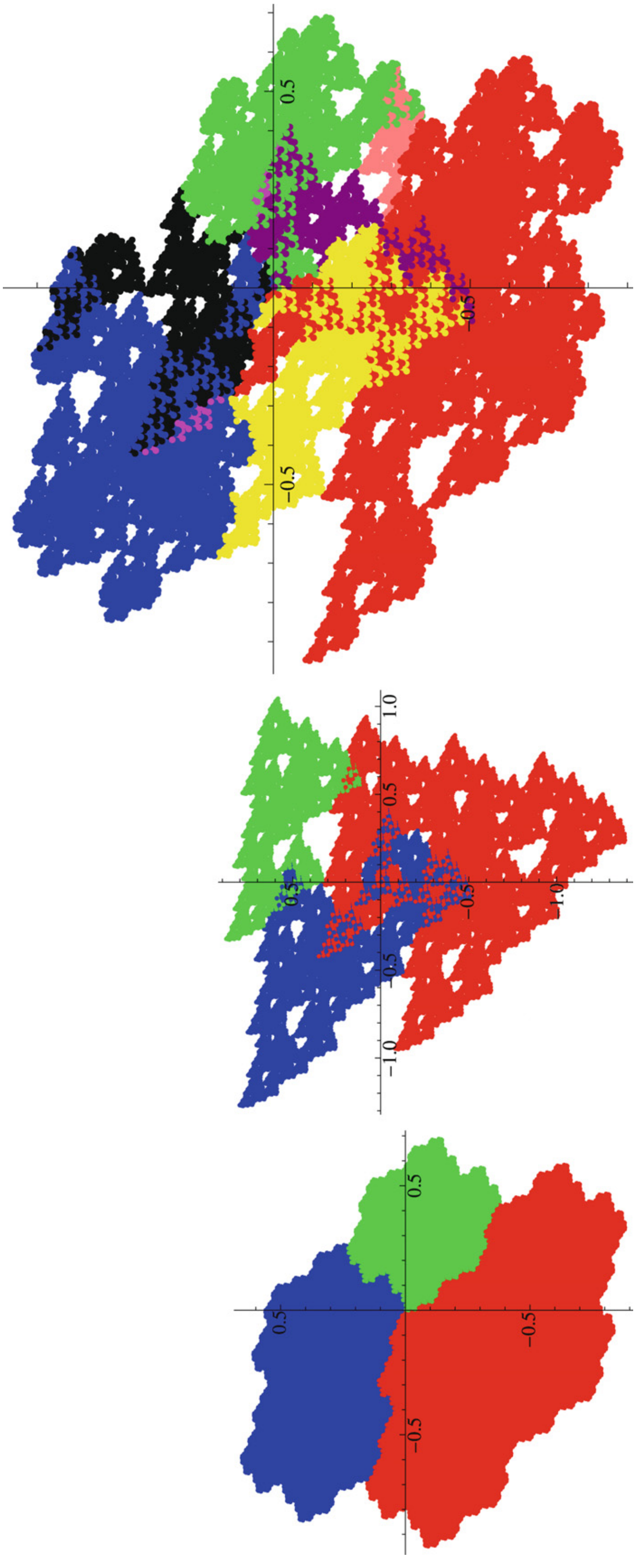

Fig. 1. Rauzy fractals of σ_1, σ_2 and their intersection.

Lemma 2.2. *Let* 0 *be an exclusive inner point of* $\mathcal{R}_1$ *the Rauzy fractal associated to an irreducible Pisot substitution. Then* $\mathcal{R}_1$ *is a fundamental domain for the projection of* Γ *on the contracting plane along the expanding line.*

Lemma 2.3. *Suppose that* 0 *is an exclusive inner point of* $\mathcal{R}_1$. *Let* W *be a non-empty open set in* $\mathcal{R}$, *define* $V_1 := \pi_1^{-1}(W) \subset \Omega_{\sigma_1}$ *and* $V_2 := \pi_2^{-1}(W) \subset \Omega_{\sigma_2}$. *For any* $y \in W$, *such that* $y = \pi_1(v_1) = \pi_2(v_2)$, *any return time of* v_2 *to* V_2 *is a return time of* v_1 *to* V_1.

Proof. We consider $v_1 \in V_1$ and $v_2 \in V_2$ such that $\pi_1(v_1) = \pi_2(v_2)$. Let n be a return time of v_2. By definition, if v is a fixed point of σ_1, $\pi_1(S^n v) = \pi_s(f(P_n))$ where P_n is a prefix of length n of v. We can extend this definition by continuity, we obtain $\pi_1(S^n v_1) = \pi_1(v_1) + \pi_s(f(P_n))$. Similar $\pi_2(S^n v_2) = \pi_2(v_2) + \pi_s(f(P'_n))$, where P'_n is a prefix of length n of v_2. Since $\pi_1(S^n v_1) = \pi_2(S^n v_2) + \pi_s[f(P'_n) - f(P_n)]$ there exists $w \in \Gamma$ such that $\pi_1(S^n v_1) = \pi_2(S^n v_2) + \pi_s(w)$.

By hypothesis $S^n v_2 \in V_2$ and $\pi_2(S^n v_2)$ is an inner point of $\mathcal{R}$. This implies that $\pi_2(S^n v_2)$ is an inner point of $\mathcal{R}_1$. Since $\pi_1(S^n v_1) \in \mathcal{R}_1$, and by hypothesis $\mathcal{R}_1$ is a fundamental domain, this means that the interior of $\mathcal{R}_1$ cannot meet $\mathcal{R}_1 + \pi_s(w)$, unless $\pi_s(w) = 0$. So we have $\pi_1(S^n v_1) = \pi_2(S^n v_2)$. Hence if n is a first return time to V_2, n is a return time to V_1. □

Definition 2.4. Let U and V be two finite words. We say that (U, V) is a balanced pair if $f(U) = f(V)$, where $f : \mathcal{A}^* \to \mathbb{Z}^d$ is the abelianization map.

Definition 2.5. A minimal balanced pair is a balanced pair (U, V) such that for every strict prefixes U_k of U and V_k of V of the same length k, $f(U_k) \neq f(V_k)$.

Lemma 2.6. *Let* u *and* v *be two fixed points of* σ_1 *and* σ_2 *respectively. We can decompose the double sequence* (u, v) *into a finite set of minimal balanced pairs.*

Proof. By definition, $\pi_1(u) = \pi_2(v) = 0$. Since Ω_{σ_2} is a minimal system and $\pi_2^{-1}(\mathcal{R}) \in \Omega_{\sigma_2}$ then the return time n_k to $\pi_2^{-1}(\mathcal{R})$ exists. From Lemma 2.3, n_k is a return time to $\pi_1^{-1}(\mathcal{R})$, which implies that $S^{n_k}(u) \in \pi_1^{-1}(\mathcal{R})$. This means that there exist two prefixes U and V of u and v respectively, such that $f(U) = f(V)$. The balanced pair (U, V) can be decomposed into minimal balanced pairs. We consider the image of each of these minimal pairs by σ_1 and σ_2. Each minimal balanced pair will appear, we consider the image of each new pair by σ_1 and σ_2 and iterate. Since (Ω_{σ_1}, S) is a minimal system, the first return time to $\pi_1^{-1}(\mathcal{R})$ is bounded, so the length of the minimal balanced pairs is bounded and all the minimal balanced pairs will appear after finite time. □

Proof of Theorem 1.3. Let us prove now that these common points can be obtained as the projection of a fixed point of a new substitution defined on the set of the minimal balanced pairs. We give an algorithm to obtain this morphism. From Lemma 2.1, there exist two finite words W_1 and W_2 prefixes of u and v respectively such that $f(W_1) = f(W_2)$. We can decompose the balanced pair (W_1, W_2) into minimal balanced pairs. Let (v_1, v_2) be the first minimal balanced pair. Then, $(\sigma_1(v_1), \sigma_2(v_2))$ is a balanced pair because σ_1 and σ_2 have the same matrix. We consider the decomposition of this balanced pair $(\sigma_1(v_1), \sigma_2(v_2))$ into minimal balanced pairs. This means that we can write $(\sigma_1(v_1), \sigma_2(v_2)) = (u_1 \cdots u_k, w_1 \cdots w_k)$ where $f(u_1) = f(w_1), \ldots, f(u_n) = f(v_n)$.

Since the set of common return times is bounded, by iteration with σ_1 and σ_2 we obtain in bounded time the set of all minimal balanced pairs. We can define the substitution $\sum$ over the finite set of minimal balanced pairs: $\sum : (U, V) \longmapsto (\sigma_1(U), \sigma_2(V))$. The set $\mathcal{R}$ is obtained as the closure of the projection of the stepped line associated to the fixed point of $\sum$. The interior is clearly substitutive with respect to the substitution $\sum$. □

Remark 1. We do not know if, under the hypotheses of the theorem, the set $\mathcal{R}_1 \cap \mathcal{R}_2$ is the closure of intersection of the interiors of $\mathcal{R}_1$ and $\mathcal{R}_2$.

Remark 2. We can give examples where the intersection is reduced to the origin, which is in this case an extremal point of the Rauzy fractal.

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