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Algebraic Geometry

The Nash problem for a toric pair and the minimal log-discrepancy

Problème de Nash pour une paire torique et la log-discrépance minimale

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ABSTRACT

This Note formulates the Nash problem for a pair consisting of a toric variety and an invariant ideal and gives an affirmative answer to the problem. We also prove that the minimal log-discrepancy is computed by a divisor corresponding to a Nash component, if the minimal log-discrepancy is finite. On the other hand there exists a Nash component such that the corresponding divisor has negative log-discrepancy, if the minimal log-discrepancy is $-\infty$.

RÉSUMÉ

Dans cette Note, nous formulons le problème de Nash pour une paire constituée d'une variété torique et d'un idéal invariant. Nous montrons que le problème admet une réponse positive. Nous montrons aussi que la log-discrépance minimale, si elle est finie, est calculée par un diviseur correspondant à une composante de Nash. D'autre part, si la log-discrépance minimale est $-\infty$, alors il existe une composante de Nash dont le diviseur correspondant est de log-discrépance négative.

1. Introduction

The Nash problem was posed by John F. Nash in his preprint (1968) which is published later as [9]. The problem is asking the bijectivity between the set of Nash components and the set of essential divisors of a singular variety X. The problem is answered positively for toric varieties and negatively in general [6]. As the counter examples are of dimension greater than 3, the Nash problem is still open for surfaces and 3-folds. The Nash problem for a surface is now steadily improving thanks to the work of M. Lejeune-Jalabert and A. Reguera-Lopez [7,8]. A Nash component is an irreducible component of the family of arcs passing through the singular locus. So it does not depend on the existence of a resolution of the singularities of X, while an essential divisor is defined by using resolutions of the singularities of X. The study of some examples gives us a feeling that we can get the information of the singularities of X from the information of the Nash components (notion without resolutions) even for the properties defined by using resolutions.

In this Note, we consider the Nash problem for a pair consisting of a variety and an ideal on the variety. Our principles are:

(i) For an object in the toric category, the Nash problem should hold;

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(ii) We should be able to see whether the singularities of the pair is log-canonical/log-terminal from information given by the Nash components.

(The first principle seems reasonable since we have some evidences [2,6,3,4]. The second principle is based on the observation for the counter example of the Nash problem [6].) We will show the principles are true for a toric pair consisting of a toric variety and an invariant ideal. When we consider a pair, the primary problem is how to formulate the Nash problem for the pair. Peter Petrov formulated the Nash problem for a toric pair and gave an affirmative answer in [10]. But his Nash components do not satisfy (ii). Our formulation of the Nash problem for a toric pair is different from his, but we use his result for our problem. Our Nash components are constructed on a modified space of X and this idea suggests a direction for the Nash problem in the general case (JSPS Grant-in-Aid No. 22340004, No. 19104001).

2. The Nash problem and minimal log-discrepancy

Definition 2.1. Let X be a scheme over an algebraically closed field k. An arc of X is a k-morphism $\alpha : \operatorname{Spec} K[\![t]\!] \to X$, where $K \supset k$ is a field extension. The space of arcs of X is denoted by X_∞ and the canonical projection $X_\infty \to X$ is denoted by π^X. For a morphism $f : Y \to X$ of k-schemes, the induced morphism between the arc spaces is denoted by $f_\infty : Y_\infty \to X_\infty$. One can find basic materials on the space of arcs in [5].

From now on we consider a pair (X, Z) consisting of a variety X over k and a closed subscheme $Z \subset X$, or equivalently $(X, \mathfrak{a})$, where $\mathfrak{a}$ is the defining ideal of Z. We always assume that $\operatorname{Sing} X \subset |Z|$.

Definition 2.2. A proper birational morphism $f : Y \to X$ with Y smooth, such that $f_{Y \setminus f^{-1}(Z)}$ is an isomorphism on $X \setminus Z$ and $f^{-1}(Z)$ is of pure codimension 1 is called a Z-resolution. When f satisfies the further conditions: $\mathfrak{a}\mathcal{O}_Y$ is invertible and $|f^{-1}(Z)|$ is of normal crossings, then it is called a log-resolution of (X, Z). A divisor over X is called Z-essential if it appears in every Z-resolution and is called log-essential if it appears in every log-resolution.

Definition 2.3. For a pair (X, Z), let $f : Y \to X$ be a Z-resolution and E_i $(i = 1, \dots, r)$ be the irreducible exceptional divisors of f. We say that E_i is a Z-Nash divisor if the closure of $f_\infty((\pi^Y)^{-1}(E_i))$ is an irreducible component of $(\pi^X)^{-1}(\operatorname{Sing} X)$ and call this component a Z-Nash component. Note that among all divisors over X there is a unique Z-Nash divisor up to birational equivalence for a fixed Z-Nash component.

Theorem 2.4. *(See Petrov [10].) Let X be an affine toric variety and Z an invariant closed subscheme. Then the set of Z-Nash divisors and the set of Z-essential divisors coincide.*

Definition 2.5. Let (X, Z) be a pair with X a normal $\mathbb{Q}$-Gorenstein variety. For a divisor E over X, the log-discrepancy of (X, Z) with respect to E is

$$a(E; X, Z) := \operatorname{ord}_E(K_{Y/X}) - \operatorname{ord}_E(Z) + 1,$$

where let E appears on a normal variety Y birational to X. The minimal log-discrepancy of (X, Z) is defined by

$$\operatorname{mld}(X, Z) = \inf\bigl\{a(E; X, Z) \bigm| E \text{ divisor over } X\bigr\}.$$

Note that if $\dim X \geqslant 2$ and $\operatorname{mld}(X, Z) < 0$, then $\operatorname{mld}(X, Z) = -\infty$. A pair (X, Z) is log-canonical (resp. log-terminal) if and only if $\operatorname{mld}(X, Z) \geqslant 0$ (resp. $\operatorname{mld}(X, Z) > 0$). For a log-canonical pair (X, Z), if $\operatorname{mld}(X, Z) = a(E; X, Z)$, then we say that E computes the minimal log-discrepancy.

The following shows that Z-Nash divisor does not necessarily compute the minimal log-discrepancy for (X, Z). The notation and terminologies on toric geometry are based on [1].

Example 1. Let X be $\mathbb{A}^3_{\mathbb{C}}$ and Z be defined by the ideal $\mathfrak{a} = (x_1^d x_2, x_2^d x_3, x_3^d x_1)$. Then, $|Z|$ is the union of x_i-axes $(i = 1, 2, 3)$. As a toric variety, X is defined by a cone $\sigma := \sum_{i=1}^3 \mathbb{R}_{\geqslant 0}\mathbf{e}_i$ in $N_{\mathbb{R}} = \mathbb{R}^3$, where $\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)$.

The Z-Nash divisors are $D_{\mathbf{p}_i}$ $(i = 1, 2, 3)$ which correspond to $\mathbf{p}_1 = (0, 1, 1), \mathbf{p}_2 = (1, 0, 1), \mathbf{p}_3 = (1, 1, 0)$. When $d = 2$, we can see that $\operatorname{mld}(X, Z) = 0$, while $a(D_{\mathbf{p}_i}; X, Z) = 1$ for $i = 1, 2, 3$. When $d = 3$, we can see that $\operatorname{mld}(X, Z) = -\infty$, while $a(D_{\mathbf{p}_i}; X, Z) = 1$ for $i = 1, 2, 3$.

In order to produce divisors which compute the minimal log-discrepancy, we need to modify X into a more reasonable space. We will see that for a toric pair (X, Z), the normalized blow up of X by the defining ideal $\mathfrak{a}$ of Z is an appropriate space.

Definition 2.6. Let (X, Z) be a toric pair and let $\varphi : \overline{X} \to X$ be the normalized blow up by the defining ideal $\mathfrak{a}$ of Z. Let $f : Y \to X$ be a log-resolution and E_i $(i = 1, \dots, r)$ be the irreducible exceptional divisors of f, then f factors as $f = \varphi \circ g$ for $g : Y \to \overline{X}$. We say that E_i is a log-Nash divisor for (X, Z) if the closure of $g_\infty((\pi^Y)^{-1}(E_i))$ is an irreducible component of $(\pi^{\overline{X}})^{-1}(\varphi^{-1}(Z))$ and call this component a log-Nash component. Note that among all divisors over X there is a unique log-Nash divisor up to birational equivalence for a fixed log-Nash component.

Theorem 2.7. *Let (X, Z) be a toric pair, then the following hold*:

(i) *The set of log-Nash divisors for (X, Z) coincides with the set of log-essential divisors*;
(ii) *If X is $\mathbb{Q}$-Gorenstein and (X, Z) is log-canonical, then a log-Nash divisor computes the minimal log-discrepancy*;
(iii) *If X is $\mathbb{Q}$-Gorenstein and (X, Z) is not log-canonical, then there is a log-Nash divisor with a negative log-discrepancy.*

Proof. First of all, note that the normalized blow up $\varphi : \overline{X} \to X$ is a toric morphism. Actually it corresponds to the decomposition into the dual fan of the Newton polygon $\Gamma_+(\mathfrak{a})$ for the ideal $\mathfrak{a}$ of Z. In the same way as in [6, Theorem 2.15], we have the first inclusion of the following, while the second one is trivial:

$$\begin{aligned}\{\text{log-Nash divisors for } (X, Z)\} &\subset \{\text{log-essential divisors for } (X, Z)\} \\ &\subset \{\text{divisors appearing in every toric log-resolution of } (X, Z)\}.\end{aligned}$$

For the statement (i), it is sufficient to show the equality of the first and the third sets. In fact, the first set is the same as $\{\varphi^{-1}(Z)$-Nash divisors$\}$ and it coincides with $\{\varphi^{-1}(Z)$-essential divisors$\}$ by Petrov's result Theorem 2.4. His proof also shows that this set is the same as {divisors appearing in every toric $\varphi^{-1}(Z)$-resolution}. As a toric log-resolution always factor through $\overline{X}$ and an invariant divisor on a non-singular toric variety is always normal crossings, therefore a toric $\varphi^{-1}(Z)$-resolution of $\overline{X}$ is the same as a toric log-resolution of (X, Z). Thus, it follows the required coincidence of the sets.

In order to prove (ii) and (iii), we remark that for a prime divisor E and an effective divisor D on a non-singular variety Y and the generic point γ of $(\pi^Y)^{-1}(E)$,

$$\operatorname{ord}_E(D) = \operatorname{ord}_\gamma\big(\mathcal{O}_Y(-D)\big).$$

Let r be the index of K_X. Let $\mathcal{L} = \varphi^* \omega_X^{[r]} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}}(r\varphi^{-1}(Z))$, then it is an invertible sheaf on $\mathcal{O}_{\overline{X}}$. If (X, Z) is log-canonical, then $\mathcal{L}$ is moreover an ideal sheaf of $\mathcal{O}_{\overline{X}}$. Indeed, on a toric log-resolution $f : Y \to X$, we have

$$a(E; X, Z) = \frac{1}{r}\big(\operatorname{ord}_E(rK_{Y/X}) - r \operatorname{ord}_E f^{-1}(Z) + r\big) = \frac{1}{r} \operatorname{ord}_E\big(-f^*(rK_X) - rf^{-1}(Z)\big) \geqslant 0 \tag{1}$$

for every invariant prime divisor E on Y. Here we used that $K_Y = -\sum_{D:\,\text{invariant prime divisor}} D$. Therefore, $-f^*(rK_X) - rf^{-1}(Z)$ is an effective Cartier divisor on Y, which yields that $\mathcal{L} = g_* g^*(\mathcal{L}) = g_*(\mathcal{O}_Y(f^*(rK_X) + rf^{-1}(Z))) \subset \mathcal{O}_{\overline{X}}$, where $g : Y \to \overline{X}$ is the factorization of f by $\varphi : \overline{X} \to X$. Now, we see by (1),

$$a(E; X, Z) = \frac{1}{r} \operatorname{ord}_\gamma\big(g^*\mathcal{L}\big) = \frac{1}{r} \operatorname{ord}_{g_\infty(\gamma)}(\mathcal{L}),$$

where γ is the generic point of $(\pi^Y)^{-1}(E)$. If E computes the minimal log-discrepancy, take the log-Nash divisor E_0 with the generic point $\alpha \in (\pi^Y)^{-1}(E_0)$ such that $g_\infty(\gamma) \in \overline{g_\infty(\alpha)}$. Then, by the upper semi-continuity of the order in the arc space,

$$\operatorname{mld}(X, Z) = \operatorname{ord}_{g_\infty(\gamma)}(\mathcal{L}) \geqslant \operatorname{ord}_{g_\infty(\alpha)}(\mathcal{L}) = a(E_0; X, Z),$$

which shows that E_0 computes the minimal log-discrepancy as required in (ii).

If (X, Z) is not log-canonical, then $\mathcal{L} \not\subset \mathcal{O}_{\overline{X}}$. Indeed, if $\mathcal{L} \subset \mathcal{O}_{\overline{X}}$, for every prime divisor E over X with the generic point $\gamma \in (\pi^Y)^{-1}(E)$, we have $a(E; X, Z) = \frac{1}{r} \operatorname{ord}_\gamma \mathcal{O}_Y(g^*\mathcal{L}) \geqslant 0$, which implies that (X, Z) is log-canonical, a contradiction. Now, we can put $\mathcal{L} = \mathcal{O}_{\overline{X}}(D - D')$, where $D > 0$ and $D' \geqslant 0$ are invariant divisors which do not have common components. Then, for a prime divisor $E \leqslant D$

$$a(E; X, Z) = \frac{1}{r} \operatorname{ord}_E\big(-D + D'\big) < 0.$$

As E is a prime divisor on $\overline{X}$ with the support in $|\varphi^{-1}(Z)|$, it is a log-Nash divisor.

Remark 1. One can prove (ii) and (iii) also by the combinatorics on the fan.

Another way to prove (ii) is to observe that every divisor that computes the minimal log-discrepancy is a log-essential divisor and then use (i). This way provides us with the stronger fact that all divisors that compute the minimal log discrepancy are log-Nash divisors. But I presented a proof above which does not use (i), because this proof may be useful to study a general case in which the Nash problem does not hold.

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