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Modeling of rod-structures in nonlinear elasticity

Modélisation des structures-poutres en élasticité non linéaire

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ABSTRACT

This Note deals with the modeling of a structure made of straight elastic rods whose thickness tends to 0. We show that, upon an adequate scaling, the infimum of the total elastic energy tends to the minimum of a functional which depends on fields defined on the centerlines of the rods.

RÉSUMÉ

Cette Note traite de la modélisation d'une structure formée de poutres droites élastiques. Nous montrons, après une normalisation convenable, que l'infimum de l'énergie élastique totale tend vers le minimum d'une fonctionnelle qui dépend de champs définis sur les axes des poutres.

Version française abrégée

Cette Note traite de la modélisation d'une structure $\mathcal{S}_\delta$ formée de poutres droites dont les sections droites sont des disques de rayon δ. Les axes des poutres forment le squelette $\mathcal{S}$. Dans un premier temps, nous introduisons une notion de déformation élémentaire de la structure qui généralise celle introduite dans [1] pour une seule poutre. Une déformation élémentaire est donnée par deux champs définis sur le squelette. Le premier $\mathcal{V}$ représente la déformation du squelette et le second $\mathbf{R}$ la rotation des sections droites. En particulier, au voisinage de chaque jonction, une déformation élémentaire est égale à une translation-rotation. Nous énonçons ensuite un théorème d'approximation d'une déformation quelconque par une déformation élémentaire (voir Théorème 1). La structure étant fixée en des extrémités, de ce théorème on déduit tout d'abord deux inégalités de type Korn non linéaires (voir Théorème 2) pour les déformations admissibles de $\mathcal{S}_\delta$ dont l'espace est noté $\mathbb{D}_\delta$. Nous sommes aussi en mesure de donner le comportement asymptotique du tenseur de Green–St Venant $E(v_\delta) = 1/2((\nabla v_\delta)^T \nabla v_\delta - \mathbf{I}_3)$ pour une suite de déformations v_δ telle que $\| \mathrm{dist}(\nabla_x v_\delta, SO(3))\|_{L^2(\mathcal{S}_\delta)} = O(\delta^\kappa)$ $(1 < \kappa \leqslant 2)$.

Nous considérons alors une structure élastique dont la densité d'énergie W est celle d'un matériau de St Venant–Kirchhoff (voir [2]) et qui est soumise à des forces volumiques $f_{\kappa,\delta}$. Pour simplifier ici la présentation, nous supposons que ces forces ne dépendent que de l'abscisse curviligne dans les poutres et sont constantes dans les jonctions. L'énergie totale est donnée par $J_{\kappa,\delta}(v) = \int_{\mathcal{S}_\delta} W(E(v)) - \int_{\mathcal{S}_\delta} f_{\kappa,\delta} \cdot (v - I_d)$ si $\det(\nabla v) > 0$ et où I_d est l' application identité. Les inégalités de Korn du Théorème 2 permettent de calibrer les forces en fonction de δ de telle sorte que l'infimum $m_{\kappa,\delta} = \inf_{v \in \mathbb{D}_\delta} J_{\kappa,\delta}(v)$ soit d'ordre $\delta^{2\kappa}$. Une suite minimisante vérifie $\| \mathrm{dist}(\nabla_x v_\delta, SO(3))\|_{L^2(\mathcal{S}_\delta)} = O(\delta^\kappa)$. Notre but est alors de caractériser la li-

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mite de la suite $\frac{m_{\kappa,\delta}}{\delta^{2\kappa}}$. Dans le théorème 3 ($1<\kappa<2$), nous montrons que cette limite est le minimum d'une fonctionnelle linéaire sur un ensemble de déformations contractantes du squelette. Dans le théorème 4 ($\kappa=2$), la limite est le minimum d'une fonctionnelle non linéaire sur un ensemble formé de déformations de type isométriques du squelette et de rotations des sections droites des poutres.

1. Introduction

This Note concerns the modeling of a structure $\mathcal{S}_\delta$ made of elastic straight rods whose cross sections are discs of radius δ. The centerlines of the rods form the skeleton structure $\mathcal{S}$. To any deformation v of $\mathcal{S}_\delta$, we associate an elementary deformation v_e which generalizes the decomposition of a large deformation of a single rod introduced in [1]. An elementary deformation is characterized by two fields defined on the skeleton $\mathcal{S}$. The first one $\mathcal{V}$ stands for the centerlines deformation while the second one $\mathbf{R}$ represents the rotations of the cross sections. In Theorem 1 we give estimates on $\mathcal{V}$, $\mathbf{R}$ and $v-v_e$ in terms of the geometrical energy $\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{S}_\delta)}$ and δ.

We then consider a St Venant–Kirchhoff elastic structure submitted to applied body forces $f_{\kappa,\delta}$. The total energy is given by $J_{\kappa,\delta}(v)=\int_{\mathcal{S}_\delta} W(E(v))-\int_{\mathcal{S}_\delta} f_{\kappa,\delta}\cdot(v-I_d)$ if $\det(\nabla v)>0$ where I_d is the identity map. We scale $f_{\kappa,\delta}$ w.r.t. δ and κ in such a way that the infimum of the total energy be of order $\delta^{2\kappa}$. In Theorems 3 and 4 we characterize the asymptotic behavior of the rescaled infimum.

The decomposition of the displacements for rod-structures has been introduced in [4,5]. As a general reference for the theory of elasticity we refer to [2]. In the framework of linear elasticity, we refer to [7] and [6] for the junction of rods. The application of Γ-convergence arguments in order to justify a nonlinear model for a single straight rod can be found in [8].

The detailed proofs will be presented in forthcoming papers.

2. The geometry of the structure

For an integer $N\geqslant 1$ and $i\in\{1,\dots,N\}$, let γ_i be a segment parametrized by s_i, with direction the unit vector $\mathbf{t}_i$, origin the point $P_i\in\mathbb{R}^3$ and length L_i. So, the generic point of γ_i is $\varphi_i(s_i)=P_i+s_i\mathbf{t}_i$, $0\leqslant s_i\leqslant L_i$. The extremities of all the segments make up a set denoted Γ.

For any $i\in\{1,\dots,N\}$, we choose a unit vector $\mathbf{n}_i$ normal to $\mathbf{t}_i$ and we set $\mathbf{b}_i=\mathbf{t}_i\wedge\mathbf{n}_i$.

The structure-skeleton is $\mathcal{S}=\bigcup_{i=1}^N\gamma_i$. The common points to two segments are the knots. Their set is $\mathcal{K}$. Among all the knots, $\Gamma_{\mathcal{K}}$ is the set of those which are extremities of all segments containing them. For any $A\in\mathcal{K}$ and all $k\in\{1,\dots,N\}$ such that $A\in\gamma_k$, we set $\overrightarrow{P_kA}=a_k\mathbf{t}_k$.

Geometrical hypothesis. *We assume the following hypotheses on* $\mathcal{S}$:

- $\mathcal{S}$ *is connected,*
- $\forall(i,j)\in\{1,\dots,N\}^2$ *if* $i\neq j$ *then* $\gamma_i\cap\gamma_j=\emptyset$ or $\gamma_i\cap\gamma_j$ *is reduced to one knot.*

Let $\delta>0$, we denote by $\Omega_{i,\delta}=]0,L_i[\times\omega_\delta$ (respectively $\Omega_i=]0,L_i[\times\omega$) the right circular cylinder of length L_i and radius δ (resp. 1). For any $i\in\{1,\dots,N\}$, the straight rod $\mathcal{P}_{i,\delta}$ is the right circular cylinder of center line γ_i and radius δ. We have $\mathcal{P}_{i,\delta}=\Phi_i(\Omega_{i,\delta})$ where $\Phi_i(s)=\varphi_i(s_i)+y_2\mathbf{n}_i+y_3\mathbf{b}_i$ is defined for $s=(s_i,y_2,y_3)\in\mathbb{R}^3$.

The whole structure is $\mathcal{S}_\delta=(\bigcup_{i=1}^N\mathcal{P}_{i,\delta})\cup(\bigcup_{A\in\Gamma_{\mathcal{K}}}B(A;\delta))$. There exists $\delta_0>0$ such that for $0<\delta\leqslant\delta_0$ and for all $(i,j)\in\{1,\dots,N\}^2$, we have $\mathcal{P}_{i,\delta}\cap\mathcal{P}_{j,\delta}=\emptyset$ if and only if $\gamma_i\cap\gamma_j=\emptyset$.

The generic point of the cylinder $\Omega_{i,\delta}$ (resp. Ω_i, $\mathcal{S}_\delta$) is $s=(s_i,y_2,y_3)$ (resp. (s_i,Y_3,Y_3), x).

3. Decomposition of a deformation

We set $L^2(\mathcal{S};\mathbb{R}^p)=\prod_{i=1}^N L^2(0,L_i;\mathbb{R}^p)$ equiped with the product norm and

$$H^1\big(\mathcal{S};\mathbb{R}^p\big)=\left\{V\in\prod_{i=1}^N H^1\big(0,L_i;\mathbb{R}^p\big)\ \Big|\ V=(V_1,\dots,V_N),\ \text{s.t.}\ \forall A\in\mathcal{K},\ \forall(i,j)\in\{1,\dots,N\}^2\right.$$
$$\left.\text{with } A\in\gamma_i\cap\gamma_j,\ \text{one has } V_i(a_i)=V_j(a_j)\right\}.$$

The common value $V_i(a_i)$ is denoted $V(A)$. We equip $H^1(\mathcal{S};\mathbb{R}^p)$ with the product norm. Notice that the parameterization of the identity map is $\varphi=(\varphi_1,\dots,\varphi_N)\in H^1(\mathcal{S};\mathbb{R}^3)$.

For any knot A and any $\rho>0$, we set $\mathcal{J}_{A,\rho\delta}=B(A;\rho\delta)\cap\mathcal{S}_\delta$. Up to choosing δ_0 small enough, there exists a real number $1\leqslant\rho_0\leqslant\frac{1}{4\delta_0}\min_{A\in\mathcal{K},B\in\mathcal{K},A\neq B}\|AB\|_2$ depending on $\mathcal{S}$ (via the angles between the segments of the skeleton $\mathcal{S}$) such that for any $\delta\in]0,\delta_0]$ the set $\mathcal{S}_\delta\setminus\bigcup_{A\in\mathcal{K}}\mathcal{J}_{A,\rho_0\delta}$ is made by disjointed cylinders. The junction in the neighborhood of A is the domain $\mathcal{J}_{A,\rho_0\delta}$.

Definition 1. An *elementary rod-structure deformation* is a deformation v_e verifying in each rod $\mathcal{P}_{i,\delta}$ ($i \in \{1, \dots, N\}$) and each junction $\mathcal{J}_{A,\rho_0\delta}$

$$v_e(s) = \mathcal{V}_i(s_i) + \mathbf{R}_i(s_i)(y_2\mathbf{n}_i + y_3\mathbf{b}_i) \quad s = (s_i, y_2, y_3) \in \Omega_{i,\delta},$$
$$v_e(x) = \mathcal{V}(A) + \mathbf{R}(A)(x - A) \quad x \in \mathcal{J}_{A,\rho_0\delta},$$

where $\mathcal{V} \in H^1(\mathcal{S}; \mathbb{R}^3)$ and $\mathbf{R} \in H^1(\mathcal{S}; SO(3))$ are such that v_e belongs to $H^1(\mathcal{S}_\delta; \mathbb{R}^3)$.

In Definition 1, the two expressions of v_e identify on each domain $\mathcal{J}_{A,\rho_0\delta} \cap \mathcal{P}_{i,\delta}$. Let us mention that the field $\mathcal{V}_i$ stands for the deformation of the line γ_i while $\mathbf{R}_i(s_i)$ represents the rotation of the cross section with arc lenght s_i on γ_i. We can prove that any deformation in $H^1(\mathcal{S}_\delta; \mathbb{R}^3)$ is approximated by an elementary rod-structure deformation.

Theorem 1. *Let v be a deformation in $H^1(\mathcal{S}_\delta; \mathbb{R}^3)$. There exists an elementary rod-structure deformation v_e in the sense of Definition* 1 *such that if we set $\bar{v} = v - v_e$*

$$\|\bar{v}\|_{L^2(\mathcal{S}_\delta;\mathbb{R}^3)} + \delta\|\nabla\bar{v}\|_{L^2(\mathcal{S}_\delta;\mathbb{R}^{3\times3})} \leqslant C\delta\big\|\mathrm{dist}\big(\nabla v, SO(3)\big)\big\|_{L^2(\mathcal{S}_\delta)}.$$

Moreover the fields $\mathcal{V}$ and $\mathbf{R}$ associated to v_e satisfy

$$\sum_{i=1}^N \left(\delta\left\|\frac{d\mathbf{R}_i}{ds_i}\right\|_{L^2(0,L_i;\mathbb{R}^{3\times3})} + \left\|\frac{d\mathcal{V}_i}{ds_i} - \mathbf{R}_i\mathbf{t}_i\right\|_{L^2(0,L_i;\mathbb{R}^3)}\right) \leqslant \frac{C}{\delta}\big\|\mathrm{dist}\big(\nabla v, SO(3)\big)\big\|_{L^2(\mathcal{S}_\delta)},$$
$$\sum_{i=1}^N \|\nabla v - \mathbf{R}_i\|_{L^2(\Omega_{i,\delta};\mathbb{R}^{3\times3})} \leqslant C\big\|\mathrm{dist}\big(\nabla v, SO(3)\big)\big\|_{L^2(\mathcal{S}_\delta)}.$$

4. Korn's type inequalities for the rod-structure

We assume that the structure $\mathcal{S}_\delta$ is clamped on a few extremities whose set is denoted by Γ_0^δ, corresponding to a set Γ_0 of extremities of $\mathcal{S}$. Then we set $\mathbb{D}_\delta = \{v \in H^1(\mathcal{S}_\delta; \mathbb{R}^3) \mid v = I_d \text{ on } \Gamma_0^\delta\}$ (I_d denotes the identity map). Theorem 1 allows us to prove the following Korn's type inequalities for the structure $\mathcal{S}_\delta$ (for a fixed domain, a nonlinear Korn's inequality is established in [3]).

Theorem 2. *For any deformation $v \in \mathbb{D}_\delta$ we have*

$$\|v - I_d\|_{H^1(\mathcal{S}_\delta;\mathbb{R}^3)} \leqslant \frac{C}{\delta}\big\|\mathrm{dist}\big(\nabla v, SO(3)\big)\big\|_{L^2(\mathcal{S}_\delta)}, \qquad \|v - I_d\|_{H^1(\mathcal{S}_\delta;\mathbb{R}^3)} \leqslant C\big\{\delta + \big\|\mathrm{dist}\big(\nabla v, SO(3)\big)\big\|_{L^2(\mathcal{S}_\delta)}\big\}.$$

5. Elastic structure

We assume that the structure $\mathcal{S}_\delta$ is submitted to body forces $f_{\kappa,\delta}$ and that its total energy is given by ($\mathbf{I}_3$ is the unit 3×3 matrix)

$$J_{\kappa,\delta}(v) = \int_{\mathcal{S}_\delta} \widehat{W}(\nabla v) - \int_{\mathcal{S}_\delta} f_{\kappa,\delta} \cdot (v - I_d), \quad \text{with } \widehat{W}(F) = \begin{cases} W(\frac{1}{2}(F^TF - \mathbf{I}_3)) & \text{if } \det(F) > 0, \\ +\infty & \text{if } \det(F) \leqslant 0, \end{cases}$$

where $W(S) = \frac{\lambda}{2}(tr(S))^2 + \mu\, tr(S^2)$ (W is the St Venant–Kirchhoff elastic density). Let us consider f in $L^2(\mathcal{S}; \mathbb{R}^3)$, $F_A \in \mathbb{R}^3$ for any $A \in \mathcal{K}$ and assume that the forces $f_{\kappa,\delta}$ are given by

$$f_{\kappa,\delta}(x) = \begin{cases} \delta^{2\kappa-3}F_A & \text{in } \mathcal{J}_{A,\rho_0\delta}, \text{ for any knot } A, \\ \delta^{2\kappa-2}f_i(s_i) & \text{in } \mathcal{P}_{i,\delta} \setminus \cup_{A\in\mathcal{K}}\mathcal{J}_{A,\rho_0\delta}, \end{cases} \quad x = \Phi_i(s),$$

which means that the forces are constant in each junction $\mathcal{J}_{A,\rho_0\delta}$. Now, from Theorem 2 and the assumptions on the body forces, we obtain

$$c\delta^{2\kappa} \leqslant m_{\kappa,\delta} = \inf_{v\in\mathbb{D}_\delta} J_{\kappa,\delta}(v) \leqslant 0.$$

Recall that generally, a minimizer of $J_{\kappa,\delta}$ does not exist on $\mathbb{D}_\delta$.

6. Asymptotic behavior of the sequence $m_{\kappa,\delta}$, $1<\kappa\leqslant 2$

The goal of this section is to establish Theorems 3 and 4. Let us first introduce a few notations. We rescale $\Omega_{i,\delta}$ using the operator $\Pi_{i,\delta}$ defined for any measurable function ψ over $\Omega_{i,\delta}$

$$\Pi_{i,\delta}(\psi)(s_i,Y_2,Y_3)=\psi(s_i,\delta Y_2,\delta Y_3)\quad \text{for almost any } (s_i,Y_2,Y_3)\in\Omega_i.$$

We denote by $\mathcal{C}$ the convex hull of the set $SO(3)$. We set[1]

$$\mathbb{B}_1=\left\{\mathcal{V}\in H^1\big(\mathcal{S};\mathbb{R}^3\big)\ \big|\ \mathcal{V}=\varphi \text{ on } \Gamma_0,\ \exists\mathbf{R}\in L^2(\mathcal{S};\mathcal{C}),\ \frac{\mathrm{d}\mathcal{V}_i}{\mathrm{d}s_i}=\mathbf{R}_i\mathbf{t}_i,\ i\in\{1,\dots,N\}\right\},$$

$$\mathbb{B}_2=\left\{(\mathcal{V},\mathbf{R})\in H^1\big(\mathcal{S};\mathbb{R}^3\big)\times H^1\big(\mathcal{S};SO(3)\big)\ \big|\ \mathcal{V}=\varphi,\ \mathbf{R}_i=\mathbf{I}_3 \text{ on } \Gamma_0,\ \frac{\mathrm{d}\mathcal{V}_i}{\mathrm{d}s_i}=\mathbf{R}_i\mathbf{t}_i,\ i\in\{1,\dots,N\}\right\}.$$

For any $\mathcal{V}\in\mathbb{B}_1$, we define the operator $\mathcal{L}(\mathcal{V})=\sum_{i=1}^N\pi\int_0^{L_i}f_i\cdot(\mathcal{V}_i-\varphi_i)+\sum_{A\in\mathcal{K}}C_AF_A\cdot(\mathcal{V}(A)-\varphi(A))$ where the constant C_A is the measure of the set $\mathcal{J}_{A,\rho_0}$.

Theorem 3. *Assume that* $1<\kappa<2$. *We have* $\lim_{\delta\to0}\frac{m_{\kappa,\delta}}{\delta^{2\kappa}}=\min_{\mathcal{V}\in\mathbb{B}_1}(-\mathcal{L}(\mathcal{V}))$.

Idea of the proof. We first show $\min_{\mathcal{V}\in\mathbb{B}_1}(-\mathcal{L}(\mathcal{V}))\leqslant\liminf_{\delta\to0}\frac{m_{\kappa,\delta}}{\delta^{2\kappa}}$. Let $(v_\delta)_\delta$ be a sequence of deformations belonging to $\mathbb{D}_\delta$ and such that $\lim_{\delta\to0}\frac{J_{\kappa,\delta}(v_\delta)}{\delta^{2\kappa}}=\liminf_{\delta\to0}\frac{m_{\kappa,\delta}}{\delta^{2\kappa}}$. From the estimate on $m_{\kappa,\delta}$, we obtain $\|\operatorname{dist}(\nabla v_\delta,SO(3))\|_{L^2(\mathcal{S}_\delta)}\leqslant C\delta^\kappa$. Using Theorem 1, we decompose v_δ. There exists a subsequence still indexed by δ, $\mathcal{V}^0$ in $\mathbb{B}_1$ and $\mathbf{R}^0$ in $L^2(\mathcal{S};\mathcal{C})$ such that

$$\mathbf{R}_\delta\rightharpoonup\mathbf{R}^0\quad\text{weakly in } L^2(\mathcal{S};\mathcal{C}),\qquad \mathcal{V}_\delta\rightharpoonup\mathcal{V}^0\quad\text{weakly in } H^1\big(\mathcal{S};\mathbb{R}^3\big).$$

Indeed $\lim_{\delta\to0}\frac{1}{\delta^{2\kappa}}\int_{\mathcal{S}_\delta}f_{\kappa,\delta}\cdot(v_\delta-I_d)=\mathcal{L}(\mathcal{V}^0)$ and then $\min_{\mathcal{V}\in\mathbb{B}_1}(-\mathcal{L}(\mathcal{V}))\leqslant-\mathcal{L}(\mathcal{V}^0)\leqslant\liminf_{\delta\to0}\frac{m_{\kappa,\delta}}{\delta^{2\kappa}}$.

To prove $\min_{\mathcal{V}\in\mathbb{B}_1}(-\mathcal{L}(\mathcal{V}))\geqslant\limsup_{\delta\to0}\frac{m_{\kappa,\delta}}{\delta^{2\kappa}}$, let $\mathcal{V}^1\in\mathbb{B}_1$, associated to $\mathbf{R}^1$, be such that $\mathcal{L}(\mathcal{V}^1)=\max_{\mathcal{V}\in\mathbb{B}_1}\mathcal{L}(\mathcal{V})$. We build a sequence of elementary deformations $v^{(n)}$ in $\mathbb{D}_\delta\cap W^{1,\infty}(\mathcal{S}_\delta;\mathbb{R}^3)$, defined by $(\mathcal{V}^{(n)},\mathbf{R}^{(n)})$ in $\mathbb{B}_2$, such that

$$\mathcal{V}^{(n)}\rightharpoonup\mathcal{V}^1\quad\text{weakly in } H^1\big(\mathcal{S};\mathbb{R}^3\big),\qquad \mathbf{R}^{(n)}\rightharpoonup\mathbf{R}^1\quad\text{weakly in } L^2(\mathcal{S};\mathcal{C}),$$
$$\big\|\nabla v^{(n)}-\mathbf{R}_i^{(n)}\big\|_{(L^\infty(\Omega_{i,\delta}))^9}\leqslant C_n\delta,\qquad \frac{1}{2\delta^{\kappa-1}}\Pi_{i,\delta}\big(\big(\nabla v^{(n)}\big)^T\nabla v^{(n)}-\mathbf{I}_3\big)\to0\quad\text{strongly in } \big(L^\infty(\Omega_i)\big)^9.$$

Hence, if δ is small enough, we get $\det(\nabla v^{(n)})>0$ a.e. in $\mathcal{S}_\delta$. Then, we show $J_{\kappa,\delta}(v^{(n)})/\delta^{2\kappa}\to-\mathcal{L}(\mathcal{V}^{(n)})$ as δ tends to 0. That gives $-\mathcal{L}(\mathcal{V}^{(n)})\geqslant\limsup_{\delta\to0}\frac{m_{\kappa,\delta}}{\delta^{2\kappa}}$. We pass to the limit as n tends to infinity, this concludes the proof of the theorem.

Theorem 4. *Let* $\mathcal{J}_2$ *be the functional defined on* $\mathbb{B}_2$ *by*

$$\mathcal{J}_2(\mathcal{V},\mathbf{R})=\frac{\pi}{4}\sum_{i=1}^N\int_0^{L_i}\big(\mu\big|\Gamma_{i,1}(\mathbf{R})\big|^2+E\big|\Gamma_{i,2}(\mathbf{R})\big|^2+E\big|\Gamma_{i,3}(\mathbf{R})\big|^2\big)-\mathcal{L}(\mathcal{V}),$$
$$\Gamma_{i,1}(\mathbf{R})=\frac{d\mathbf{R}_i}{ds_i}\mathbf{n}_i\cdot\mathbf{R}_i\mathbf{b}_i,\qquad \Gamma_{i,2}(\mathbf{R})=\frac{d\mathbf{R}_i}{ds_i}\mathbf{b}_i\cdot\mathbf{R}_i\mathbf{t}_i,\qquad \Gamma_{i,3}(\mathbf{R})=\frac{d\mathbf{R}_i}{ds_i}\mathbf{t}_i\cdot\mathbf{R}_i\mathbf{n}_i,$$

where E is the Young modulus. Then we have $\lim_{\delta\to0}\frac{m_{2,\delta}}{\delta^4}=\min_{(\mathcal{V},\mathbf{R})\in\mathbb{B}_2}\mathcal{J}_2(\mathcal{V},\mathbf{R})$.

Idea of the proof. *Step* 1. In this step we show $\min_{(\mathcal{V},\mathbf{R})\in\mathbb{B}_2}\mathcal{J}_2(\mathcal{V},\mathbf{R})\leqslant\liminf_{\delta\to0}\frac{m_{2,\delta}}{\delta^4}$. Let $(v_\delta)_\delta$ be a sequence of admissible deformations such that $\lim_{\delta\to0}\frac{J_{\kappa,\delta}(v_\delta)}{\delta^4}=\liminf_{\delta\to0}\frac{m_{2,\delta}}{\delta^4}$. From Theorems 1 and 2 we obtain $\|\operatorname{dist}(\nabla v_\delta,SO(3))\|_{L^2(\mathcal{S}_\delta)}+\|\nabla v_\delta^T\nabla v_\delta-\mathbf{I}_3\|_{L^2(\mathcal{S}_\delta;\mathbb{R}^{3\times3})}\leqslant C\delta^2$. We decompose v_δ as in Theorem 1. There exists a subsequence still indexed by δ such that

$$\mathbf{R}_\delta\rightharpoonup\mathbf{R}^0\quad\text{weakly in } H^1\big(\mathcal{S};SO(3)\big),\qquad \mathcal{V}_\delta\to\mathcal{V}^0\quad\text{strongly in } H^1\big(\mathcal{S};\mathbb{R}^3\big),$$
$$\frac{1}{\delta}\bigg(\frac{d\mathcal{V}_{i,\delta}}{ds_i}-\mathbf{R}_{i,\delta}\mathbf{t}_i\bigg)\rightharpoonup\mathcal{Z}_i^0\quad\text{weakly in } \big(L^2(0,L_i)\big)^3,\qquad \frac{1}{\delta^2}\Pi_{i,\delta}(\bar v_\delta)\rightharpoonup\bar v_i^0\quad\text{weakly in } \big(L^2\big(0,L_i;H^1(\omega)\big)\big)^3,$$
$$\frac{1}{2\delta}\Pi_{i,\delta}\big((\nabla v_\delta)^T\nabla v_\delta-\mathbf{I}_3\big)\rightharpoonup\mathbf{E}_i^0\quad\text{weakly in } L^2\big(\Omega_i;\mathbb{R}^{3\times3}\big),\ i\in\{1,\dots,N\},$$

[1] We can alternatively define $\mathbb{B}_1$ as $\mathbb{B}_1=\{\mathcal{V}\in H^1(\mathcal{S};\mathbb{R}^3)\mid\mathcal{V}=\varphi \text{ on } \Gamma_0,\ \|\frac{\mathrm{d}\mathcal{V}_i}{\mathrm{d}s_i}\|_2\leqslant1,\ i\in\{1,\dots,N\}\}$. The first definition appears more natural in view of the proof of Theorem 3.

where

$$\mathbf{E}_i^0 = \left(\frac{d\mathbf{R}_i^0}{ds_i}(Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3) \;\middle|\; \frac{\partial \bar{u}_i^0}{\partial Y_2} \;\middle|\; \frac{\partial \bar{u}_i^0}{\partial Y_3} \right)^T \mathbf{R}_i^0 + \left(\mathbf{R}_i^0\right)^T \left(\frac{d\mathbf{R}_i^0}{ds_i}(Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3) \;\middle|\; \frac{\partial \bar{u}_i^0}{\partial Y_2} \;\middle|\; \frac{\partial \bar{u}_i^0}{\partial Y_3} \right)$$

with $\bar{u}_i^0 = [Y_2(\mathcal{Z}_i^0 \cdot \mathbf{R}_i^0\mathbf{n}_i) + Y_3(\mathcal{Z}_i^0 \cdot \mathbf{R}_i^0\mathbf{b}_i)]\mathbf{t}_i + (\mathbf{R}_i^0)^T \bar{v}_i^0$. The couple $(\mathcal{V}^0, \mathbf{R}^0)$ belongs to $\mathbb{B}_2$. We show that $\liminf_{\delta\to 0} \frac{1}{\delta^4} \times \int_{\mathcal{S}_\delta} \widehat{W}(\nabla v_\delta) \geqslant \sum_{i=1}^N \int_{\Omega_i} W(\mathbf{E}_i^0)$ so that $\liminf_{\delta\to 0} \frac{m_{2,\delta}}{\delta^4} \geqslant \sum_{i=1}^N \int_{\Omega_i} W(\mathbf{E}_i^0) - \mathcal{L}(\mathcal{V}^0)$. Then, minimizing w.r.t. to the $\bar{u}_i^0$'s leads to the result.

Step 2. Now we show that $\limsup_{\delta\to 0} \frac{m_{2,\delta}}{\delta^4} \leqslant \min_{(\mathcal{V},\mathbf{R})\in\mathbb{B}_2} \mathcal{J}_2(\mathcal{V}, \mathbf{R})$. Let $(\mathcal{V}^2, \mathbf{R}^2) \in \mathbb{B}_2$ be such that $\mathcal{J}_2(\mathcal{V}^2, \mathbf{R}^2) = \min_{(\mathcal{V},\mathbf{R})\in\mathbb{B}_2} \mathcal{J}_2(\mathcal{V}, \mathbf{R})$. For $i \in \{1, \dots, N\}$, let $\bar{v}_i$ be arbitrary in $W^{1,\infty}(\Omega_i; \mathbb{R}^3)$.

We build a sequence $v_\delta \in \mathbb{D}_\delta \cap W^{1,\infty}(\mathcal{S}_\delta; \mathbb{R}^3)$ such that $\det(\nabla v_\delta(x)) > 0$ for a.e. $x \in \mathcal{S}_\delta$ and whose elementary part is an approximation of $(\mathcal{V}^2, \mathbf{R}^2)$ and which satisfies

$$\begin{aligned} \frac{1}{2\delta} \Pi_{i,\delta}\big((\nabla v_\delta)^T \nabla v_\delta - \mathbf{I}_3\big) \to \mathbf{E}_i^2 = {} & \left(\frac{d\mathbf{R}_i^2}{ds_i}(Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3) \;\middle|\; \frac{\partial \bar{v}_i}{\partial Y_2} \;\middle|\; \frac{\partial \bar{v}_i}{\partial Y_3} \right)^T \mathbf{R}_i^2 \\ & + \left(\mathbf{R}_i^2\right)^T \left(\frac{d\mathbf{R}_i^2}{ds_i}(Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3) \;\middle|\; \frac{\partial \bar{v}_i}{\partial Y_2} \;\middle|\; \frac{\partial \bar{v}_i}{\partial Y_3} \right) \quad \text{strongly in } L^2\big(\Omega_i; \mathbb{R}^{3\times 3}\big). \end{aligned}$$

For this specific sequence, we prove that $\limsup_{\delta\to 0} \frac{J_{2,\delta}(v_\delta)}{\delta^4} \leqslant \sum_{i=1}^N \int_{\Omega_i} W(\mathbf{E}_i^2) - \mathcal{L}(\mathcal{V}^2)$. Minimizing the right hand side of the previous inequality w.r.t. to the $\bar{v}_i$'s we obtain the result. This concludes the proof of the theorem.

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