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# Non-oriented solutions of the eikonal equation

## *Solutions non orientées de l'équation eikonale*

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ABSTRACT

We study a new formulation for the eikonal equation $|\nabla u| = 1$ on a bounded subset of $\mathbb{R}^2$. Instead of a vector field $\nabla u$, we consider a field $P$ of orthogonal projections on one-dimensional subspaces, with $\operatorname{div} P \in L^2$. We prove that solutions of this equation propagate direction as in the classical eikonal equation. We also show that solutions exist if and only if the domain is a tubular neighborhood of a regular closed curve.



RÉSUMÉ

Nous étudions une nouvelle formulation de l'équation eikonale $|\nabla u| = 1$ sur un sous-ensemble borné de $\mathbb{R}^2$. Au lieu d'un champ de vecteurs $\nabla u$, nous considérons un champ $P$ de projections orthogonales sur les sous-espaces de dimension 1, avec $\operatorname{div} P \in L^2$. Nous montrons que les solutions de cette équation propagent la direction comme dans l'équation eikonale classique. Nous montrons aussi que les solutions existent si et seulement si le domaine est un voisinage tubulaire d'une courbe régulière fermée.



### 1. Stripe patterns and the eikonal equation

Many pattern-forming systems produce parallel stripes, both straight and curved. In this Note we report on a new mathematical description of curved striped patterns. We recently studied the behavior of a stripe-forming energy, and investigated a limit process in which the stripe width tends to zero [4]. In that limit the stripes not only become thin, but also uniform in width, and the stripe pattern comes to resemble the level sets of a solution of the eikonal equation. The rigorous version of this statement, in the form of a Gamma-convergence result, gives rise to a new formulation of the eikonal equation, in which the directionality of the stripes is represented, rather than by vector fields, by *line fields*, which capture direction only up to a sign (Fig. 1 (right)).

The line field is represented by a *projection*, which for the purposes of this paper we define to be a matrix $P$ that can be written in terms of a unit vector $m$ as $P = m \otimes m$. Note that both $m$ and $-m$ give rise to the same projection $P$; this is the *unsigned* nature of a line field.

The *projection-valued eikonal equation* is as follows. Let $\Omega$ be an open subset of $\mathbb{R}^2$. Find $P \in L^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})$, with $P = 0$ a.e. in $\mathbb{R}^2 \backslash \Omega$, such that

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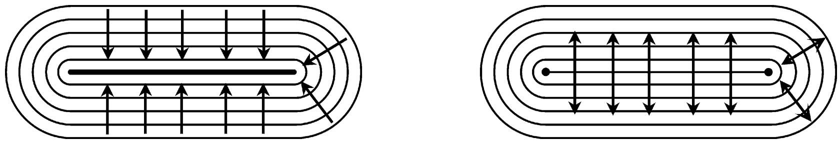

**Fig. 1.** Stripe patterns represented by level sets of functions $u$ with $|\nabla u| = 1$ (left, arrows are the gradient $\nabla u$) or by unoriented line fields (right).

$$P^2 = P, \qquad \operatorname{rank}(P) = 1, \qquad P \text{ is symmetric} \quad \text{a.e. in } \Omega, \tag{1a}$$

$$\operatorname{div} P \in L^2(\mathbb{R}^2; \mathbb{R}^2), \tag{1b}$$

$$P \operatorname{div} P = 0 \qquad \text{a.e. in } \Omega. \tag{1c}$$

Here $\operatorname{div} P$ is defined as the vector-valued function whose $i$-th component is given by $(\operatorname{div} P)_i := \sum_{j=1}^{2} \partial_{x_j} P_{ij}$.

The first line encodes the property that $P(x)$ is a projection, in the sense above, at a.e. $x \in \Omega$. Given the regularity provided by (1b), the final condition (1c) is the eikonal equation itself, as a calculation for a smooth unit-length vector field $m(x)$ shows. Indeed, we have

$$0 = P \operatorname{div} P = m\big(m \cdot (m \operatorname{div} m + \nabla m \cdot m)\big) = m \operatorname{div} m + m(m \cdot \nabla m \cdot m) = m \operatorname{div} m, \tag{2}$$

where the final equality follows from differentiating the identity $|m|^2 = 1$. A solution vector field $m$ therefore is divergence-free, implying that its rotation over 90 degrees is a gradient $\nabla u$; from $|m| = 1$ follows the classical eikonal equation $|\nabla u| = 1$.

Property (1b) also prevents the normal component of $P$ from jumping across $\partial\Omega$, and it implies that $P \cdot n = 0$ in the sense of traces on $\partial\Omega$.

The exponent 2 in (1b) is critical in the following sense. Natural possibilities for singularities in a line field are jump discontinuities ('grain boundaries') and target patterns. At a grain boundary the jump in $P$ causes $\operatorname{div} P$ to have a line singularity, comparable to the one-dimensional Hausdorff measure; condition (1b) clearly excludes that possibility. For a target pattern the curvature $\kappa$ of the stripes scales as $1/r$, where $r$ is the distance to the center; then $\int \kappa^p$ is locally finite for $p < 2$, and diverges logarithmically for $p = 2$. The cases $p < 2$ and $p \geqslant 2$ therefore distinguish between whether target patterns are admissible ($p < 2$) or not.

## 2. Results

Our first result shows that solutions of (1), as in the case of other eikonal-equation formulations, preserve directional information in the normal direction:

**Theorem 2.1.** *Let $\Omega$ be an open, bounded, and connected subset of $\mathbb{R}^2$ with $C^2$ boundary, and let $P$ be a solution of* (1). *Then*

$$P \in H^1(\Omega; \mathbb{R}^{2\times 2}), \qquad P \cdot n = 0 \quad \text{a.e. on } \partial\Omega. \tag{3}$$

*Moreover, let $x_0$ be a Lebesgue point of $P$ in $\Omega$, let $x \in \Omega$, and let $L$ be the line segment connecting $x_0$ with $x$. Assume that $L \subset \Omega$. If $P(x_0) \cdot (x - x_0) = 0$, then $P(y) = P(x_0)$ for $\mathcal{H}^1$-almost every $y \in L$.*

**Proof.** Statement (3) can be obtained by algebraic manipulation of system (1) (we omit this calculation). The proof of the propagation of direction is an application of the classical characteristics principle for vector-field solutions of the eikonal equation. The first step is therefore to construct a *lifting* $m$ such that $P = m \times m$, and this is done by the following lemma by Ball and Zarnescu:

**Lemma 2.2.** *(See [1].) Let $\Omega$ be an open, bounded, connected subset of $\mathbb{R}^2$, with $C^2$ boundary. If $P$ satisfies* (1) *and has the additional regularity $P \in H^1(\Omega; \mathbb{R}^{2\times 2})$, then there exists a vector field $m \in H^1(\Omega; S^1)$ such that*

$$P = m \otimes m \quad \text{a.e. on } \Omega,$$

*if and only if there exists a vector field $\overline{m} \in H^{1/2}(\partial\Omega; S^1)$, such that*

$$\operatorname{Tr}(P)_{|\partial\Omega} = \overline{m} \otimes \overline{m} \quad \text{a.e. on } \partial\Omega.$$

Owing to the boundary condition in (3), for a.e. $x \in \partial\Omega$, $P(x)$ is the orthogonal projection on the line tangent to $\partial\Omega$ in $x$. We can then construct a lifting $\overline{m}$ of $P$ on the boundary simply by taking the derivatives of any smooth arclength parametrization of $\partial\Omega$. Since such a vector field satisfies $\operatorname{Tr}(P)_{|\partial\Omega} = \overline{m} \otimes \overline{m}$, by Lemma 2.2 there exists a vector field $m \in H^1(\Omega, \mathbb{R}^2)$ such that $P = m \otimes m$. Note that $P \cdot n = 0$ on the boundary also implies $m \cdot n = 0$, a.e. on $\partial\Omega$. Let $m^\perp$ be the 90 degrees clockwise rotation of $m$. The vector field $m$ satisfies

$$\begin{aligned} &\big|m(x)\big| = 1 && \text{for a.e. } x \text{ in } \Omega, && (4)\\ &\operatorname{div} m = 0 && \text{distributionally in } \mathbb{R}^2, && (5)\\ &\nabla m \cdot m^\perp = 0 && \text{a.e. in } \Omega. && (6) \end{aligned}$$

Property (4) follows from remarking that $|m|^4 = m \cdot P \cdot m = m \cdot P^2 \cdot m = |m|^6$. The computation (2) yields (5), and any $m \in H^1(\Omega)$ satisfying (4) and (5) also satisfies the characteristics principle (6), which implies that $m$, and therefore $P$, is constant in the normal direction. □

The second main result shows that the restrictions on $P$ are so rigid that the mere existence of a solution provides a strong characterization of the geometry of the domain $\Omega$:

**Theorem 2.3.** *Let $\Omega$ be an open, bounded, and connected subset of $\mathbb{R}^2$ with $C^2$ boundary. Then there exists a solution of* (1) *if and only if $\Omega$ is a tubular domain. In that case the solution is unique.*

A *tubular domain* is a domain in $\mathbb{R}^2$ that can be written as $\Omega = \Gamma + B(0,\delta)$, where $\Gamma$ is a closed curve in $\mathbb{R}^2$ with continuous and bounded curvature $\kappa$, $0 < \delta < \|\kappa\|_\infty^{-1}$, and $B(0,\delta)$ is the open ball of center 0 and radius $\delta$.

**Proof of Theorem 2.3.** The non-trivial part of Theorem 2.3 is to show that existence of a solution $P$ implies that $\Omega$ is tubular.

The vector field $m^\perp$ is orthogonal to $\partial\Omega$ and by property (5) it satisfies $\operatorname{curl}(m^\perp) = 0$, therefore by Green's theorem there exists a potential $\phi \in H^2(\Omega)$ such that $\nabla\phi = m^\perp$, $\phi$ is constant on every connected component of $\partial\Omega$, and by (6) and (4), $\phi$ is linear with slope 1 on the normal lines to the boundary. By the regularity of $\phi$ we deduce that the normal lines to $\partial\Omega$ cannot intersect and therefore, by the regularity of the boundary it is possible to build a continuous fibration of $\Omega$ through segments which are orthogonal to $\partial\Omega$ and have uniform length $\max_\Omega \phi - \min_\Omega \phi$. This concludes the proof of Theorem 2.3. □

## 3. Discussion

The work of this Note represents a first step in the analysis of this projection-valued eikonal equation. While the main results are still lacking in various ways — see below — the main point of this paper is to show that this projection-valued formulation is a useful alternative to the usual vector-based formulation.

To start with, our Theorems 2.1 and 2.3 show that solutions of (1) behave much like we expect from the eikonal equation, in the sense that directional information is preserved in the normal direction. Theorem 2.3 makes this property even more explicit, by showing that a full tube, or bunch, of parallel 'stripes' can be identified.

However, it is the differences with the vector-valued eikonal equation that are the most interesting. Fig. 1 shows how this formulation can be a better representation of the physical reality than the vector-based form. On the left, the vector field has a jump discontinuity along the center line, while on the right the projection is continuous along that line. Depending on the underlying model, this singularity may have a physical counterpart, or may be a spurious consequence of the vector-based description. For a wave-propagation model the singularity is very real; for striped-pattern systems it typically is not. A projection-valued formulation therefore provides an alternative to the Riemann-surface approach that is sometimes used [2].

We now comment in more detail on our method of proof. The proof of the properties that we give in this paper relies on a reduction of the projection-valued formulation to a vector-based formulation. This reduction is achieved by the Ball–Zarnescu lemma (Lemma 2.2), which requires $\operatorname{div} P \in L^2$; for less regularity the existence of a lifting may not hold, as the example of a U-turn pattern shows.

The dependence of the proof on a vector-based lifting is awkward in various ways. To start with, the condition $\operatorname{div} P \in L^2$ required for the lifting is much stronger than the conditions (4), (5), and a weak formulation of (6), which are required in [3] for the conservation of information in the normal direction. It also has the effect of excluding all singularities, as we already remarked. It would be interesting to prove properties such as those of Theorems 2.1 and 2.3 by methods that do not rely on this lifting.

We would hope that such an intrinsic projection-based proof could also be generalized to the study of target patterns and U-turns, and eventually of grain boundaries. These will require increasingly weak regularity requirements: target patterns may exist for $\operatorname{div} P \in L^p$ with $p < 2$, and for a line discontinuity, such as a grain boundary, $\operatorname{div} P$ will be a measure.

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