Mathematical Problems in Mechanics

Penalized direct forcing and projection schemes for Navier–Stokes

Méthode de forçage direct pénalisé et schémas de projection pour Navier–Stokes

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A R T I C L E   I N F O

A B S T R A C T

This Note presents a new method of direct forcing to deal with obstacles in incompressible flows. It mixes projection schemes and velocity L^2 penalty schemes. The penalized direct forcing term is distributed in the velocity prediction and the correction equations. It leads to a natural treatment in the correction equation of the boundary conditions in pressure around obstacles. A numerical experiment provided an illustration of the method.

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R É S U M É

Cette Note présente une nouvelle méthode de forçage directe pour prendre en compte des obstacles dans un écoulement incompressible. Elle mélange méthodes de projection et de pénalisation L^2 des vitesses. La ventilation du terme de forçage direct pénalisé dans les équations de prédiction et de correction conduit à un traitement naturel des conditions aux limites pour la correction de pression aux bords des obstacles. Une expérience numérique est présentée à titre d'illustration.

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1. Introduction

The well-known incompressible Navier–Stokes equations allow the description of the behavior of incompressible flows (div $\mathbf{u} = 0$ with $\mathbf{u}$ the fluid velocity). Mass and momentum balance equations in the fluid domain Ω_f read as (with Dirichlet boundary conditions – BC –, $\mathbf{u} = \mathbf{u}_D$ on $\partial \Omega_f$):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \otimes \nabla \mathbf{u} + \rho^{-1} \nabla p - \mathrm{div}(\nu \nabla \mathbf{u}) = \mathbf{f} & \text{in } \Omega_f, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega_f \end{cases} \tag{1}$$

with p the pressure, ρ the (constant) density, ν the cinematic viscosity and $\mathbf{f}$ the volume force.

The projection method (or fractional step method) for an incompressible fluid was introduced by Chorin and Temam in 1968 [2,10]. The incremental projection scheme of Shen [9] reads as follow.[1] First, a non-divergence free velocity $\tilde{\mathbf{u}}$ is predicted using the previous time step pressure. For example, a linearized implicit Euler time discretization,[2] with $\mathbf{u}^{n,*}$ given by an Adam–Bashforth scheme, leads to:

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[1] Projection schemes for dilatable or barotropic fluid can be found in [4,6].

[2] Using the Crank–Nicholson time scheme should be worth trying, but it is not the goal of this Note.

$$\frac{\tilde{\mathbf{u}} - \mathbf{u}^n}{\Delta t} + \mathbf{u}^{n,*} \otimes \nabla \tilde{\mathbf{u}} + \rho^{-1} \nabla p^n - \text{div}(\nu \nabla \tilde{\mathbf{u}}) = \mathbf{f}^{n+1} \quad \text{in } \Omega_f \quad \text{and} \quad \tilde{\mathbf{u}} = \mathbf{u}_D \quad \text{on } \partial \Omega_f. \tag{2}$$

Second, with the hypothesis $\text{div}(\nu \nabla \tilde{\mathbf{u}}) \sim \text{div}(\nu \nabla \mathbf{u}^{n+1})$ and $\mathbf{u}^{n,*} \otimes \nabla \tilde{\mathbf{u}} \sim \mathbf{u}^{n,*} \otimes \nabla \mathbf{u}^{n+1}$, a correction stage is performed to recover the divergence free velocity $\mathbf{u}^{n+1}$:

$$\begin{cases} \frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}}{\Delta t} = -\rho^{-1} \nabla \phi^{n+1} & \text{in } \Omega_f \quad \text{and} \quad \mathbf{u}^{n+1} = \mathbf{u}_D \quad \text{on } \partial \Omega_f \\ \text{div} \, \mathbf{u}^{n+1} = 0 & \text{in } \Omega_f \end{cases} \tag{3}$$

with $\phi^{n+1} = p^{n+1} - p^n$ the pressure correction. Considering the divergence free condition, we are able to compute ϕ^{n+1} and so on the divergence free velocity $\mathbf{u}^{n+1}$ and the new pressure p^{n+1} by:

$$\text{div} \, \nabla \phi^{n+1} = \frac{\rho}{\Delta t} \text{div} \, \tilde{\mathbf{u}} \quad \text{in } \Omega_f \quad \text{and} \quad \nabla \phi^{n+1} \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega_f. \tag{4}$$

2. Direct forcing

In the context of the Fictitious Domain methods, the Immersed Boundary method was introduced by Peskin at the beginning of 1970 [8]. Considering obstacles in a flow, the fluid domain Ω_f is extended to Ω including the obstacle domain Ω_s: $\Omega = \Omega_f \cup \Omega_s$. A body force is added to the momentum equation to take into account the obstacle. In 1997 Mohd-Yusof introduced the Direct Forcing method [7]:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \mathbf{T}_{n,n+1} + \mathbf{F}_{n,n+1} \quad \text{in } \Omega \tag{5}$$

with $\mathbf{T}_{n,n+1} = \mathbf{f}^{n+1} - \mathbf{u}^{n,*} \otimes \nabla \mathbf{u}^{n+1} - \rho^{-1} \nabla p^{n+1} + \text{div}(\nu \nabla \mathbf{u}^{n+1})$. The forcing term is defined as:

$$\mathbf{F}_{n,n+1} = \chi_s \left[\frac{(\mathbf{v}_s^{n+1} - \mathbf{u}^n)}{\Delta t} - \mathbf{T}_{n,n+1} \right] \tag{6}$$

with $\mathbf{v}_s$ the imposed fluid velocity around/inside the obstacle and $\chi_s : R^3 \to R$ the discrete characteristic function or the volume ratio function of the obstacle, see [3].

Remark 1. The imposed velocity $\mathbf{v}_s$ needs to be interpolated from the fluid velocity field $\tilde{\mathbf{u}}$ and the boundary condition to improve the accuracy ($\mathbf{v}_s = \bar{\varepsilon} \tilde{\mathbf{u}}$, see Fadlun et al. [3]).

Remark 2. The direct forcing should be taken into account into the projection (4) through the interpolation matrix $\bar{\varepsilon}$ ($\text{div} \, \bar{\varepsilon} \nabla \phi = \frac{\rho}{\Delta t} \text{div} \, \tilde{\mathbf{u}}$) to get a free divergence velocity respecting the boundary conditions (consistent scheme, see Ikeno et al. [5]).

3. Penalized direct forcing

In this new method, the following forcing term is added to the right-hand side of Navier–Stokes equations:

$$\mathbf{F}_{n+1} = \frac{\chi_s}{\eta \Delta t} \left(\mathbf{v}_s^{n+1} - \mathbf{u}^{n+1} \right) \quad \text{with } 0 < \eta \ll 1. \tag{7}$$

It can be viewed as an implicit forcing term $c \frac{\chi_s}{\Delta t} (\mathbf{v}_s - \mathbf{u})$ as well as a velocity L^2 penalty term $c \frac{\chi_s}{\eta} (\mathbf{v}_s - \mathbf{u})$ [1]. As for the Direct Forcing, interpolations can be used to determine the imposed velocity $\mathbf{v}_s$. In the following, $\mathbf{u}^{n+1}$ is the solution of penalized Navier–Stokes. Lets define $\mathbf{u}_\eta^{n+1} := \lim_{\eta \to 0} \mathbf{u}^{n+1}$. Then, we have $\mathbf{u}_\eta^{n+1} \sim \mathbf{v}_s^{n+1}$. Following Angot et al. [1], $\mathbf{u}_\eta^{n+1}$ converges toward the N.-S. body fitted solution in $L^2(\Omega_s)$ norm with order $\leqslant 3/4$ in η. The new feature of our algorithm is that the forcing term (7) is distributed in the prediction and the correction stages of the projection leading to a natural consistent scheme:

$$\begin{cases} \frac{\tilde{\mathbf{u}} - \mathbf{u}^n}{\Delta t} + \mathbf{u}^{n,*} \otimes \nabla \tilde{\mathbf{u}} + \rho^{-1} \nabla p^n - \text{div} \, \nu \nabla \tilde{\mathbf{u}} = \mathbf{f}^{n+1} + \frac{\chi_s}{\eta \Delta t} \left(\mathbf{v}_s^{n+1} - \tilde{\mathbf{u}} \right) & \text{in } \Omega, \\ \frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}}{\Delta t} = -\rho^{-1} \nabla \phi^{n+1} + \frac{\chi_s}{\eta \Delta t} \left(\tilde{\mathbf{u}} - \mathbf{u}^{n+1} \right) & \text{in } \Omega. \end{cases} \tag{8}$$

The first equation of (8) leads to the penalized predicted velocity: $\tilde{\mathbf{u}}_\eta := \lim_{\eta \to 0} \tilde{\mathbf{u}} \sim \mathbf{v}_s^{n+1}$ in Ω_s. Hence the Dirichlet boundary conditions of (2) are locally enforced on $\partial \Omega_f \cap \partial \Omega_s$. The second equation of (8) suggests:

$$\frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}}{\Delta t} = -\check{\rho}^{-1} \nabla \phi^{n+1} \quad \text{in } \Omega \text{ with suitable BC on } \partial \Omega. \tag{9}$$
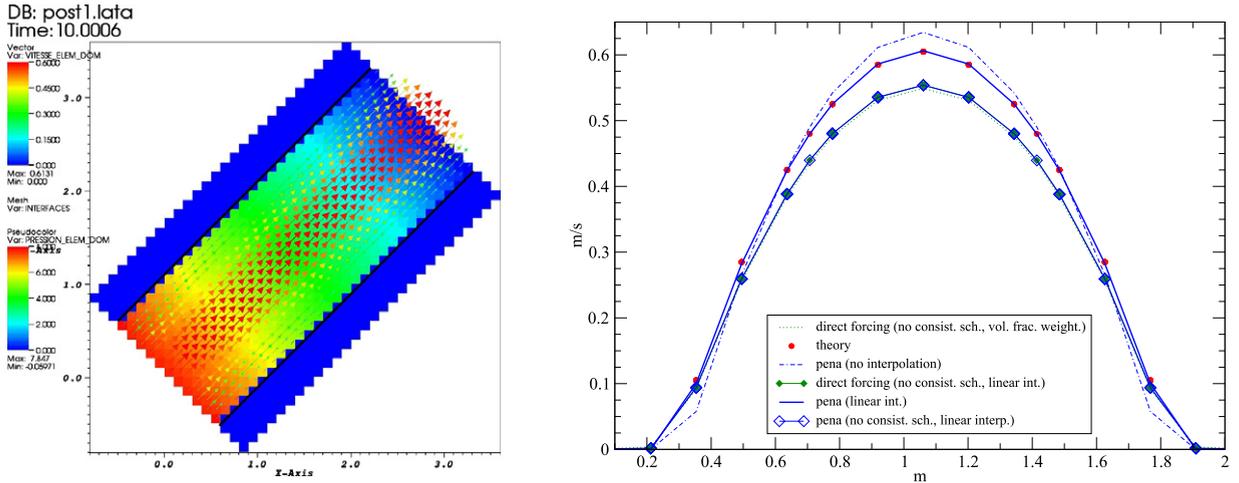
Fig. 1. Poiseuille flow: direct forcing and penalized direct forcing. Left: velocity and pressure fields (linearly interpolated Penalized Direct Forcing). Right: velocity profiles (no interpolation or linear interpolation, no consistent or consistent scheme).

This equation is similar to the first one of (3) with $\check{\rho} := \rho(1 + \frac{\chi_s}{\eta})$ justifying the introduction of the term Δt in (7). For $\chi_s = 0$, we have $\check{\rho} = \rho$ and $\lim_{\eta \to 0} \check{\rho}(\eta) \approx \mathcal{O}(\eta^{-1})$ for $\chi_s > 0$. Using $\operatorname{div} \mathbf{u}^{n+1} = 0$ in Ω_f and $\operatorname{div} \mathbf{u}^{n+1} = \operatorname{div} \mathbf{v}_s^{n+1}$ in Ω_s, instead of the second equation of (3), we get a slightly modification of the first equation of (4):

$$\operatorname{div} \frac{\eta}{\eta + \chi_s} \nabla \phi^{n+1} = \frac{\rho}{\Delta t} \big(\operatorname{div} \tilde{u} - \chi_s \operatorname{div} \mathbf{v}_s^{n+1} \big) \quad \text{in } \Omega \text{ with suitable BC on } \partial \Omega. \tag{10}$$

Lets remark that, in case of solid rotations and/or translations, $\operatorname{div} \mathbf{v}_s = 0$ in Ω_s. For $\chi_s = 1$, we have an *effective diffusion* coefficient $\frac{\eta}{1+\eta} \approx \mathcal{O}(\eta) \ll 1$. Then, the homogeneous Neumann boundary conditions of (4) are locally enforced on $\partial \Omega_f \cap \partial \Omega_s$ in a natural way. Finally, the correction (9) allows the computation of $\mathbf{u}^{n+1}$. We have $\mathbf{u}_\eta^{n+1} \sim \tilde{\mathbf{u}}_\eta \sim \mathbf{v}_s^{n+1}$ in Ω_s.

Remark 3. In Ω_s, all the diffusion coefficients may vanish leading to numerical problems solving (10). Then, adding a pressure L^2 penalty term can be a cure, leading to a different formulation of the pressure correction equation (where ϕ_0 is a prescribed pressure correction):

$$\operatorname{div} \mathcal{O}(\eta) \nabla \phi^{n+1} + \frac{1}{\eta} \big(\phi^{n+1} - \phi_0 \big) = \frac{\rho}{\Delta t} \big(\operatorname{div} \tilde{u} - \operatorname{div} \mathbf{v}_s^{n+1} \big) \quad \text{in } \Omega_s. \tag{11}$$

4. Numerical experiments

Fig. 1, left, shows a 2D Poiseuille flow in a square channel rotated of $45°$ with regard to the Cartesian mesh. Immersed boundaries define the solid walls (black lines). Imposed near-wall velocities are linearly interpolated from the nearest free fluid velocities and the boundary conditions $\mathbf{v}_s^{n+1} = \bar{\varepsilon} \mathbf{u}^n$. Computations are done using the CEA CFD code Trio_U [11]. The value of the penalty coefficient η is 10^{-12}. Velocity profiles, obtained by the linearly interpolated Penalized Direct Forcing method, compare very well to the theoretical ones, see Fig. 1, right. Without consistent schemes the theoretical velocities are missed whatever the interpolation is. Moreover, the numerical order of the method remains about 2 in L^2 norm when using a second order discretization for the spatial operators.

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