



Partial Differential Equations/Optimal Control

Boundary null-controllability of linear diffusion–reaction equations

Contrôlabilité frontière à zéro des équations linéaires de type diffusion–réaction

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ABSTRACT

This Note deals with the boundary null-controllability of linear diffusion–reaction equations in a 2D bounded domain. We transform the determination of the sought HUM boundary control into the minimization of a continuous and strictly convex functional. In the case of a rectangular domain where the diffusion tensor is represented by a diagonal matrix, we establish a procedure based on the inner product method that uses a complete orthonormal family of Sturm–Liouville's eigenfunctions to express explicitly the sought control.

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RÉSUMÉ

Il s'agit de la contrôlabilité frontière à zéro des équations linéaires de type diffusion–réaction dans un domaine borné de \mathbb{R}^2 . Nous transformons la détermination du contrôle de type HUM en la minimisation d'une fonctionnelle continue et strictement convexe. Dans le cas d'un domaine rectangulaire où le tenseur de diffusion est représenté par une matrice diagonale, nous exprimons explicitement le contrôle recherché dans une base orthonormée construite par les fonctions propres d'un problème de Sturm–Liouville.

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In the literature, to compute the HUM control authors use generally an infinite matrix as a representation of the obtained controllability operator. Although the direct method is the most used to determine the involved matrix, see [1,2,5,9], the inner product method constitutes a second option that offers some important advantages such that the no need of computing any solution to the control system, deducing the control at a new final time using the already computed control and due to the symmetry, we only need to compute half of the entries, see [10]. In this Note, we address the boundary null-controllability of linear diffusion–reaction equations. We establish the computation of the sought HUM boundary control for the general setting. Then, using the inner product method, we determine explicitly the sought control in the case of a 2D rectangular domain where the longitudinal and transversal diffusion axes coincide with the Cartesian x - and y -axes which means that the diffusion tensor is represented by a diagonal matrix.

Let Ω be a bounded connected open set of \mathbb{R}^2 with a Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$. For a fixed $T > 0$, we consider the following evolution problem governed by a diffusion–reaction equation:

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$$\begin{aligned}
 \partial_t \varphi - \operatorname{div}(D \nabla \varphi) + \rho \varphi &= 0 && \text{in } Q = \Omega \times (0, T), \\
 \varphi(\cdot, 0) &= \varphi_0 && \text{in } \Omega, \\
 \varphi &= \psi && \text{on } \sum_D = \Gamma_D \times (0, T), \\
 D \nabla \varphi \cdot \nu &= 0 && \text{on } \sum_N = \Gamma_N \times (0, T),
 \end{aligned} \tag{1}$$

where ρ is a positive real number that represents the reaction coefficient, ν is the unit normal vector exterior to $\partial\Omega$ and D is the diffusion tensor given by a 2×2 real symmetric matrix. It is well known, see, for example, [11,12], that given an initial data $\varphi_0 \in H^1(\Omega)$ and $\psi \in L^2(0, T; \Gamma_D)$, the problem (1) admits a unique solution φ such that $\varphi \in L^2(0, T; H^2(\Omega)) \cap C^0(0, T; H^1(\Omega))$ and $\partial_t \varphi \in L^2(0, T; L^2(\Omega))$.

Boundary null-controllability problem. The problem with which we are concerned here is given $\varphi_0 \in H^1(\Omega)$, find a control $\psi \in L^2(0, T; \Gamma_D)$ such that the solution φ to (1) satisfies $\varphi(\cdot, T) = 0$ in Ω .

As introduced by J.L. Lions [6,7], the Hilbert Uniqueness Method (HUM) defines the control ψ from the solution to the adjoint problem associated to (1) defined for a given initial data $v_0 \in H^1(\Omega)$ by

$$\begin{aligned}
 -\partial_t v - \operatorname{div}(D \nabla v) + \rho v &= 0 && \text{in } Q = \Omega \times (0, T), \\
 v(\cdot, T) &= v_0 && \text{in } \Omega, \\
 v &= 0 && \text{on } \sum_D = \Gamma_D \times (0, T), \\
 D \nabla v \cdot \nu &= 0 && \text{on } \sum_N = \Gamma_N \times (0, T).
 \end{aligned} \tag{2}$$

Then, inspired by [10] we introduce the so-called linear complementary boundary operator C as follows:

$$\langle C[v], \psi \rangle_{L^2(\Gamma_D)} = \langle \operatorname{div}(D \nabla \varphi) - \rho \varphi, v \rangle + \langle \varphi, -\operatorname{div}(D \nabla v) + \rho v \rangle, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product. That leads to define the boundary operator C such that $C[v] = -D \nabla v \cdot \nu$. The following lemma gives a necessary and sufficient condition on a function ψ to be an admissible control:

Lemma 1. *Let $T > 0$ and $\varphi_0 \in H^1(\Omega)$ be given. The solution $\varphi \in L^2(0, T; H^2(\Omega))$ to the problem (1) with a control $\psi \in L^2(0, T; \Gamma_D)$ satisfies $\varphi(\cdot, T) = 0$ in Ω if and only if*

$$\langle \psi, C[v] \rangle_{L^2(\sum_D)} + \langle \varphi_0, v(\cdot, 0) \rangle_{L^2(\Omega)} = 0, \tag{4}$$

for all $v \in L^2(0, T; H^2(\Omega))$ solution to the adjoint problem (2) with an initial data $v_0 \in H^1(\Omega)$.

Proof. In view of (1)–(3), it is easy to see using Green’s formula that we have

$$\langle \langle \varphi, v \rangle_{L^2(\Omega)} \rangle_0^T = \int_0^T (\langle \partial_t \varphi, v \rangle + \langle \varphi, \partial_t v \rangle) dt = \langle \psi, C[v] \rangle_{L^2(\sum_D)}. \quad \square \tag{5}$$

To determine the HUM control, we introduce the bilinear form $\gamma : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ such that

$$\gamma(v_0, z_0) = \langle C[v], C[z] \rangle_{L^2(\sum_D)} = \int_0^T \int_{\Gamma_D} D \nabla v \cdot \nu D \nabla z \cdot \nu d\Gamma dt, \tag{6}$$

where z is the solution to (2) with $z(\cdot, T) = z_0 \in H^1(\Omega)$, and the functional $J : H^1(\Omega) \rightarrow \mathbb{R}$ as follows:

$$J(v_0) = \frac{1}{2} \gamma(v_0, v_0) + \int_{\Omega} \varphi_0 v(\cdot, 0) d\Omega. \tag{7}$$

Theorem 2. *Let $\hat{v}_0 \in H^1(\Omega)$ be a minimizer of J introduced in (7) and \hat{v} be the solution to (2) with $\hat{v}(\cdot, T) = \hat{v}_0$. The HUM control $\psi = C[\hat{v}] = -D \nabla \hat{v} \cdot \nu$ solves the boundary null-controllability problem.*

Proof. Setting $\psi = C[\hat{v}]$ in the first order optimality condition, we conclude using Lemma 1. \square

Moreover, using similar techniques as done in [10], we prove that the functional J introduced in (7) admits a unique minimizer \hat{v}_0 . In order to compute the HUM control $\psi = C[\hat{v}]$, we employ the following linear operators [10]: $L_T : H^1(\Omega) \rightarrow$

$H^1(\Omega)$ such that, to a given φ_0 , associates $L_T(\varphi_0) = \varphi(\cdot, T)$ the solution to (1) computed with $\psi = 0$ and taken at $t = T$. Then, we also introduce $L_T^* : H^1(\Omega) \rightarrow H^1(\Omega)$ that to a given v_0 associates $L_T^*(v_0) = v(\cdot, 0)$ the solution to (2) taken at $t = 0$. Notice that using (5) with $\psi = 0$ gives

$$\langle \varphi(\cdot, T), v_0 \rangle_{L^2(\Omega)} = \langle \varphi_0, v(\cdot, 0) \rangle_{L^2(\Omega)} \Leftrightarrow \langle L_T(\varphi_0), v_0 \rangle_{L^2(\Omega)} = \langle \varphi_0, L_T^*(v_0) \rangle_{L^2(\Omega)}, \tag{8}$$

which implies that L_T and L_T^* are two adjoint operators. In a similar way, we introduce the operators [10]: $G_T : H^1(\Omega) \rightarrow L^2(\Sigma_D)$ such that, to a given v_0 , associates $G_T(v_0) = C[v]$ where v is the solution to (2) with $v(\cdot, T) = v_0$ and $G_T^* : L^2(\Sigma_D) \rightarrow H^1(\Omega)$ such that, to a given control ψ , associates $G_T^*(\psi) = \varphi(\cdot, T)$ the solution to (1) computed with $\varphi_0 = 0$. Therefore, in view of (5) we find

$$\langle G_T^*(\psi), v_0 \rangle = \langle \varphi(\cdot, T), v(\cdot, T) \rangle = \langle \psi, G_T(v_0) \rangle. \tag{9}$$

Thus, G_T and G_T^* are also two adjoint operators. Let $\hat{v}_0 \in H^1(\Omega)$ be the minimizer of the functional J introduced in (7). Then, according to the first order optimality condition, we obtain for all $v_0 \in H^1(\Omega)$,

$$\begin{aligned} \langle \nabla J(\hat{v}_0), v_0 \rangle &= \langle G_T(\hat{v}_0), G_T(v_0) \rangle_{L^2(\Sigma_D)} + \langle \varphi_0, L_T^*(v_0) \rangle_{L^2(\Omega)}, \\ &= \langle G_T^* G_T(\hat{v}_0), v_0 \rangle_{L^2(\Omega)} + \langle L_T(\varphi_0), v_0 \rangle_{L^2(\Omega)} = 0. \end{aligned} \tag{10}$$

Hence, the minimizer \hat{v}_0 of the functional J satisfies

$$G_T^* G_T(\hat{v}_0) = -L_T(\varphi_0) \quad \text{where } G_T^* G_T : H^1(\Omega) \rightarrow H^1(\Omega). \tag{11}$$

Inner product method. Let $(e_i)_{i \geq 0}$ be an orthonormal basis of $H^1(\Omega)$ and A be an infinite matrix that represents the controllability operator $G_T^* G_T$ introduced in (11). Then, using initial data v_0 and z_0 such that $v_0 = e_i$ and $z_0 = e_j$, we find according to the bilinear form γ introduced in (6) that $\gamma(e_i, e_j) = \langle G_T^* G_T(e_i), e_j \rangle = \langle Ae_i, e_j \rangle$ which leads to define the entries of the matrix A as follows:

$$A_{ij} = \langle Ae_i, e_j \rangle = \int_0^T \int_{\Gamma_D} D \nabla v_i \cdot \nu D \nabla v_j \cdot \nu, \quad \text{for all } i \geq 0 \text{ and } j \geq 0, \tag{12}$$

where v_i and v_j are the solutions to (2) with initial data e_i and e_j . In the remainder, we consider the case of a rectangular domain $\Omega = (0, L) \times (0, \ell)$. Here, we denote Γ_D the left-side boundary of Ω which coincides with the y -axis and $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Furthermore, the diffusion tensor D is taken to be a 2×2 diagonal matrix with entries $D_{11} > 0$ and $D_{22} \geq 0$. Then, the orthonormal family $(e_i)_{i \geq 0}$ such that for all $i \geq 0$,

$$e_i(x, y) = c_i \sin\left((2i + 1)\frac{\pi}{2L}x\right) \cos\left(i\frac{\pi}{\ell}y\right), \quad \text{where } c_i = \begin{cases} \frac{2}{\sqrt{\ell L}} & \text{if } i > 0 \\ \sqrt{\frac{2}{\ell L}} & \text{if } i = 0 \end{cases} \tag{13}$$

solves the following Sturm–Liouville’s problem:

$$\begin{aligned} -\operatorname{div}(D \nabla e_i) + \rho e_i &= \mu_i e_i & \text{in } \Omega, \\ e_i &= 0 & \text{on } \Gamma_D, \\ D \nabla e_i \cdot \nu &= 0 & \text{on } \Gamma_N, \end{aligned} \tag{14}$$

where for all $i \geq 0$, the real number $\mu_i = \rho + D_{11}((2i + 1)\pi/2L)^2 + D_{22}(i\pi/\ell)^2$ denotes the eigenvalue associated to the eigenfunction e_i . Then, by expressing v_i the solution to (2) with initial data $v_0 = e_i$ in the complete orthonormal family $(e_j)_{j \geq 0}$, we find $v_i(x, y, t) = e^{-\mu_i(T-t)} e_i(x, y)$. In addition, as the unit normal vector is $\nu = (-1, 0)^\top$ on Γ_D , we obtain according to (12) that for all $i, j \geq 0$,

$$A_{ij} = D_{11}^2 \int_0^T e^{-(\mu_i + \mu_j)(T-t)} dt \int_0^\ell \partial_x e_i(0, y) \partial_x e_j(0, y) dy = \begin{cases} \frac{D_{11} \pi (2i+1)^2}{4L^3 \mu_i} (1 - e^{-2\mu_i T}) & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Besides, in order to determine the right-hand side $-L_T(\varphi_0) = -\varphi(\cdot, T)$ involved in (11), we need to compute φ the solution to (1) with $\psi = 0$. Thus, using the complete orthonormal family $(e_i)_{i \geq 0}$, we find

$$\varphi(x, y, t) = \sum_{i \geq 0} \langle \varphi_0, e_i \rangle e^{-\mu_i t} e_i(x, y), \quad \text{in } Q = \Omega \times (0, T). \tag{16}$$

Hence, according to (15), (16) and using $\hat{v}_0 = \sum_{i \geq 0} \hat{v}_0^i e_i$, the linear system $A \hat{v}_0 = -L_T(\varphi_0)$ gives

$$\hat{v}_0^i = -\frac{e^{-\mu_i T}}{A_{ii}} \langle \varphi_0, e_i \rangle = -\frac{4L^3 \mu_i e^{-\mu_i T}}{(D_{11}\pi(2i+1))^2(1-e^{-2\mu_i T})} \langle \varphi_0, e_i \rangle, \quad \text{for all } i \geq 0. \quad (17)$$

Furthermore, the solution to the adjoint problem (2) with $\hat{v}(\cdot, T) = \hat{v}_0$ is defined by

$$\hat{v}(x, y, t) = \sum_{i \geq 0} \hat{v}_0^i e^{-\mu_i(T-t)} e_i(x, y) \quad \text{in } Q = \Omega \times (0, T). \quad (18)$$

Therefore, since the unit normal vector is $\nu = (-1, 0)^T$ on Γ_D , we obtain $\psi(y, t) = D_{11} \partial_x \hat{v}(0, y, t)$ which in view of (17) and (18) leads to define the sought HUM boundary control on $\Sigma_D = \Gamma_D \times (0, T)$ as follows:

$$\psi(y, t) = -\frac{2L^2}{D_{11}\pi} \sum_{i \geq 0} \frac{c_i \mu_i e^{-\mu_i(2T-t)}}{(2i+1)(1-e^{-2\mu_i T})} \langle \varphi_0, e_i \rangle \cos\left(i \frac{\pi}{\ell} y\right). \quad (19)$$

In [8], some numerical experiments concerning the case of a rectangular domain with a diffusion tensor given by a 2×2 diagonal matrix are presented. Those experiments show that in this particular case, the boundary null-controllability problem is well solved using the HUM boundary control derived in (19). Moreover, this numerical study is to appear involved in recent results regarding an inverse source problem that consists of the identification of pollution sources in surface water, see [3,4].

Besides, studying the extension of this technic in order to derive explicitly the HUM boundary control for different geometries of the domain Ω and a more general diffusion tensor is a work in progress.

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