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Mixed formulation for Stokes problem with Tresca friction

Formulation mixte pour le problème de Stokes avec frottement de Tresca

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A B S T R A C T

In this Note we propose and study a three field mixed formulation for solving the Stokes problem with Tresca-type nonlinear boundary conditions. Two Lagrange multipliers are used to enforce $\mathrm{div}(u) = 0$ constraint and to regularize the energy functional. The resulting problem is discretized using *P1 bubble/P1-P1* finite elements. Optimal error estimate is derived and a numerical validation test is achieved.

R É S U M É

Dans cette note, nous proposons une formulation mixte à trois champs pour résoudre le problème de Stokes avec des conditions aux limites non-linéaires, du type Tresca. Deux multiplicateurs de Lagrange sont utilisés afin d'imposer $\mathrm{div}(u) = 0$ et de régulariser la fonctionnelle énergie. Le problème résultant est approché à l'aide des éléments finis *P1 bulle/P1-P1*. Une estimation optimale est obtenue et un test numérique de validation est réalisé.

1. Introduction

The no-slip hypothesis at a fluid–wall interface leads to good agreement with experimental observations for Newtonian fluids which is no longer true for non-Newtonian fluid [6]. For example, in the flow of certain high molecular weight linear polymers through circular dies, the exit flow rate has been found to be a discontinuous function of pressure drop over a certain range of shear rates [3]. This observation is consistent with the hypothesis that the velocity at the wall is not zero. Several studies have been made and showed not only that slip takes place when a threshold is reached but also it's the origin of many defects and instabilities in the polymer injection process [8].

Many papers were published simulating various flows with such boundary conditions (see [7] and references therein). Recently, based on the penalty method, error estimates for the Stokes problem with Tresca boundary conditions with strong regularity assumption on the velocity field are obtained [5].

The aim of this work is to contribute to the numerical analysis of Stokes problem with Tresca boundary conditions. Our first purpose is to carry out the convergence analysis and a priori estimates for the mixed finite element formulation of the above cited problem. The second one is to derive an algorithm well adapted to this formulation and easy to implement in order to validate our theoretical estimates.

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The Note is organized as follows. First, we introduce the equations modeling the Stokes problem. Then we establish the continuous mixed variational formulation is Section 3. The following section is devoted to a priori error estimates, we show an optimal order of $h^{3/4}$ with $\mathbf{H}^2(\Omega)$ assumption regularity on the velocity. In Section 5 we validate the theoretical estimate by a numerical test.

2. Setting Stokes problem with nonlinear boundary conditions

We consider the following Stokes problem with nonlinear boundary conditions of Tresca friction type:

$$\begin{cases} -\mathrm{div}\big(\nu\varepsilon(\mathbf{u})\big)+\nabla p=\mathbf{f} & \text{in } \Omega,\\ \mathrm{div}(\mathbf{u})=0 & \text{in } \Omega,\\ \mathbf{u}=\mathbf{0} & \text{on } \Gamma_0,\\ \mathbf{u}_n=\mathbf{0} & \text{on } \Gamma,\\ |\sigma_t|<g \quad\Rightarrow\quad \mathbf{u}_t=\mathbf{0} & \text{on } \Gamma,\\ |\sigma_t|=g \quad\Rightarrow\quad \exists k>0 \ \text{ a constant such that } \ \mathbf{u}_t=-k\sigma_t & \text{on } \Gamma, \end{cases} \tag{1}$$

with $\Omega\subset\mathbb{R}^2$ an open set with regular boundary $\partial\Omega$, which is the union of two non-overlapping portions Γ_0 and Γ, Γ_0 is subjected to no-slip boundary conditions while Γ is where the fluid may slip. The symbol $\varepsilon(\mathbf{u})$ represents the linearized strain rate tensor $\varepsilon(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+\nabla^t\mathbf{u})$. We denote by $\mathbf{n}$ the outward unit normal to $\partial\Omega$ and $\mathbf{u}_n$ and $\mathbf{u}_t$, the normal, the tangential, component of $\mathbf{u}$ respectively. The stress vector σ is equal to $\underline{\underline{\sigma}}.\mathbf{n}$ where $\underline{\underline{\sigma}}$ is the Cauchy stress tensor defined by:

$$\underline{\underline{\sigma}}=2\nu\varepsilon(\mathbf{u})-p\underline{\underline{\delta}},$$

where p is the hydrostatic pressure, $\underline{\underline{\delta}}$ is the identity tensor and ν is the kinematic fluid viscosity.

One can derive the variational formulation of (1):

$$\begin{cases} \text{Find } \mathbf{u}\in\mathbf{V}_{div}(\Omega) \text{ such that:} \quad \forall\mathbf{v}\in\mathbf{V}_{div}(\Omega)\\ a(\mathbf{u},\mathbf{v}-\mathbf{u})+j(\mathbf{v})-j(\mathbf{u})\geqslant L(\mathbf{v}-\mathbf{u}), \end{cases} \tag{2}$$

with

$$\mathbf{V}(\Omega)=\big\{\mathbf{v}\in\mathbf{H}^1(\Omega),\ \mathbf{v}_{|\Gamma_0}=0,\ \mathbf{v}.\mathbf{n}_{|\Gamma}=0\big\},\qquad \mathbf{V}_{div}(\Omega)=\big\{\mathbf{v}\in\mathbf{V}(\Omega),\ \mathrm{div}(\mathbf{v})=0 \text{ in } \Omega\big\}$$

which are endowed by the norm: $\|\mathbf{u}\|_1=(\sum_{i=1}^2\|u_i\|^2_{H^1(\Omega)})^{1/2}$.

$$a(\mathbf{u},\mathbf{v})=\int_\Omega \nu\varepsilon(\mathbf{u})\colon\varepsilon(\mathbf{v})\,\mathrm{d}\Omega,\qquad L(\mathbf{v})=\int_\Omega \mathbf{f}\,\mathbf{v}\,\mathrm{d}\Omega,\qquad g \text{ is a non-negative function in } L^2(\Gamma)$$

and

$$j(\mathbf{v})=\int_\Gamma g|\mathbf{v}_t|\,\mathrm{d}\Gamma\quad \forall\mathbf{u},\mathbf{v}\in\mathbf{V}_{div}(\Omega)\quad \text{where } |\cdot| \text{ is the Euclidian norm in } \mathbb{R}^2.$$

Problem (2) has a unique solution [1]. Moreover, since the bilinear form $a(\cdot,\cdot)$ is symmetric, (2) is equivalent to the following constrained non-differentiable minimization problem:

$$\begin{cases} \text{Find } \mathbf{u}\in\mathbf{V}_{div}(\Omega) \text{ such that:}\\ \mathcal{J}(\mathbf{u})\leqslant\mathcal{J}(\mathbf{v})\quad \forall\mathbf{v}\in\mathbf{V}_{div}(\Omega), \end{cases} \tag{3}$$

where $\mathcal{J}(\mathbf{v})=\frac{1}{2}a(\mathbf{v},\mathbf{v})+j(\mathbf{v})-L(\mathbf{v})$.

3. Mixed formulation

Problem (3) can be written as a saddle point problem which has a unique solution characterized by

$$\begin{cases} \text{Find } \big(\mathbf{u},(p,\lambda)\big)\in\mathbf{V}(\Omega)\times\Lambda \text{ such that:}\\ a(\mathbf{u},\mathbf{v})+b\big((p,\lambda),\mathbf{v}\big)=L(\mathbf{v})\quad \forall\mathbf{v}\in\mathbf{V}(\Omega),\\ b\big((q-p,\mu-\lambda),\mathbf{u}\big)\leqslant 0\quad \forall(q,\mu)\in\Lambda, \end{cases} \tag{4}$$

where

$$b\big((p,\lambda),\mathbf{v}\big)=-(p,\mathrm{div}\,\mathbf{v})+\langle\lambda,\mathbf{v}_t\rangle\quad \text{and}\quad \Lambda=L_0^2(\Omega)\times\mathcal{Q}, \tag{5}$$

with

$$\mathcal{Q}=\left\{\mu\in\left(L^2(\Gamma)\right),\ |\mu|\leqslant g\right\}\quad\text{and}\quad L_0^2(\Omega)=\left\{p\in L^2(\Omega),\ \int_\Omega p(\mathbf{x})\,\mathrm{d}\mathbf{x}=0\right\}.$$

Lemma 3.1. *(See [1].) There exists a constant $\alpha>0$ such that*: $\forall(q,\mu)\in L_0^2(\Omega)\times L^2(\Gamma)$

$$\sup_{\mathbf{v}\in\mathbf{V}}\frac{b((q,\mu),\mathbf{v})}{\|\mathbf{v}\|_1}\geqslant\alpha\left(\|q\|_{L^2(\Omega)}+\|\mu\|_{-\frac{1}{2}}\right).\tag{6}$$

Note that $H^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H^{\frac{1}{2}}(\Gamma)=\{\varphi\in L^2(\Gamma)$ such that $\|\varphi\|_{\frac{1}{2},\Gamma}<+\infty\}$ which are equipped respectively by the norms

$$\|\psi\|_{\frac{1}{2},\Gamma}=\left(\|\psi\|_{0,\Gamma}+\int_\Gamma\int_\Gamma\frac{(\psi(\mathbf{x})-\psi(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^2}\,\mathrm{d}\Gamma\mathbf{x}\,\mathrm{d}\Gamma\mathbf{y}\right)^{\frac{1}{2}},\qquad\|\mu\|_{-\frac{1}{2},\Gamma}=\sup_{\varphi\in H^{\frac{1}{2}}(\Gamma),\varphi\neq0}\frac{\langle\mu,\varphi\rangle}{\|\varphi\|_{\frac{1}{2},\Gamma}}.$$

Theorem 3.2. *(See [1].) Suppose that $a(\cdot,\cdot)$ is continuous, V-elliptic bilinear form on $\mathbf{V}(\Omega)$ and* (6) *holds. Then there exists a unique $(\mathbf{u},(p,\lambda))$ solution of mixed problem* (4)*.*

4. Error estimates

The present section is devoted to finite element approximation of the solution of problem (4). We use classical *P1 bubble-P1* finite element to disretize $(\mathbf{u},p)$ and *P1* finite element on Γ for the Lagrange multiplier λ.

The domain Ω is supposed to be polygonal. Let $\mathcal{T}_h$ be a regular partition of $\overline{\Omega}$ with triangles. We denote by $\mathbb{P}_n(\kappa)$ the space of polynomials of degree less and equal to $n\in\mathbb{N}$ defined on $\kappa\in\mathcal{T}_h$ and by $\mathcal{B}_\kappa$ the space of bubble functions defined on κ which is a sub-space of $H_0^1(\kappa)$. Then we can define the following discrete spaces: $\mathbb{B}=\bigoplus_{\kappa\in\mathcal{T}_h}\mathcal{B}_\kappa$, $\mathcal{V}_h=\{\mathbf{v}_h\in\mathcal{C}^0(\overline{\Omega});\ \mathbf{v}_{h|\kappa}\in\mathbb{P}_1\ \forall\kappa\in\mathcal{T}_h,\ \mathbf{v}_{h|\Gamma_0}=0,$ and $\mathbf{v}_h.\mathbf{n}_{|\Gamma}=0\}$, $\mathbf{V}_h=[\mathcal{V}_h+\mathbb{B}]^2$ and $\mathcal{W}_h=\{\mathbf{v}_{h|\Gamma},\ \mathbf{v}_h\in\mathbf{V}_h\}$. Let

$$\mathbb{L}_h=\left\{q_h\in\mathcal{C}^0(\overline{\Omega});\ q_{h|\kappa}\in\mathbb{P}_1\ \forall\kappa\in\mathcal{T}_h,\ \int_\Omega q_h=0\right\},$$

$\mathcal{M}_h=\mathbb{L}_h\times\mathcal{W}_h$,

$$\mathcal{Q}_h=\left\{\mu_h\in\mathcal{W}_h,\ \int_\Gamma\mu_h\psi_h-\int_\Gamma g|\psi_h|\leqslant0\ \forall\psi_h\in\mathcal{W}_h\right\}$$

and $\Lambda_h=\mathbb{L}_h\times\mathcal{Q}_h$.

Note that $\mathcal{Q}_h$ is an external approximation of $\mathcal{Q}$, so the discretization is non-conforming.

Discretizing (4) we obtain

$$\begin{cases}\text{Find }\left(\mathbf{u}_h,(p_h,\lambda_h)\right)\in\mathbf{V}_h\times\Lambda_h\text{ such that:}\\ a(\mathbf{u}_h,\mathbf{v}_h)+b\left((p_h,\lambda_h),\mathbf{v}_h\right)=L(\mathbf{v}_h)\quad\forall\mathbf{v}_h\in\mathbf{V}_h,\\ b\left((q_h-p_h,\mu_h-\lambda_h),\mathbf{u}_h\right)\leqslant0\quad\forall(q_h,\mu_h)\in\Lambda_h,\end{cases}\tag{7}$$

where Λ_h is a closed convex of $\mathbb{L}_h\times\mathcal{W}_h$.

A sufficient condition for the existence and uniqueness of the solution to problem (7) is the *inf-sup* condition.

Proposition 4.1. *(See [1].) There exists a constant $\beta>0$ independent of h such that*:

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{b((q_h,\mu_h),\mathbf{v}_h)}{\|\mathbf{v}_h\|_1}\geqslant\beta\left(\|q_h\|_{L^2(\Omega)}+\|\mu_h\|_{-\frac{1}{2}}\right)\quad\forall(q_h,\mu_h)\in\mathbb{L}_h\times\mathcal{W}_h.\tag{8}$$

Lemma 4.2. *(See [2].) Let $(\mathbf{u},p,\lambda)$ and $(\mathbf{u}_h,p_h,\lambda_h)$ be solutions to* (4), (7) *respectively. Then for any $(\mathbf{v}_h,q_h,\mu_h)\in\mathbf{V}_h\times\Lambda_h$ it holds*:

$$\begin{aligned}a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h)\leqslant{}&a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{v}_h)+b\left((p,\lambda)-(q_h,\mu_h),\mathbf{u}_h-\mathbf{u}\right)+b\left((p,\lambda)-(p_h,\lambda_h),\mathbf{u}-\mathbf{v}_h\right)\\&+b\left((p,\lambda)-(q_h,\mu_h),\mathbf{u}\right)+b\left((p_h,\lambda_h)-(p,\lambda),\mathbf{u}\right).\end{aligned}\tag{9}$$

Table 1
Convergence rates with respect to h.

h	$\|\mathbf{u}-\mathbf{u}_h\|_0$	α_0	$\|\mathbf{u}-\mathbf{u}_h\|_1$	α_1	$\|p-p_h\|_0$	α_p
1.2500e–03	2.7779e–04	1.225	7.6057e–03	0.729	1.7985e–01	0.256
1.e–03	1.8160e–04	1.247	5.9045e–03	0.742	1.4513e–01	0.279
9.0909e–04	1.5163e–04	1.255	5.3128e–03	0.747	1.3040e–01	0.290
7.6923e–04	1.0930e–04	1.272	4.6551e–03	0.748	1.0512e–01	0.314
5.8824e–04	6.5507e–05	1.295	4.1820e–03	0.736	5.8219e–02	0.382
5.5556e–04	5.8187e–05	1.301	3.6802e–03	0.747	4.9130e–02	0.402

Theorem 4.3. *(See [1].) Let* $(\mathbf{u}, p, \lambda)$ *and* $(\mathbf{u}_h, p_h, \lambda_h)$ *be solutions to* (4), (7) *respectively. Suppose that* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *and* $p \in H^1(\Omega)$*. Then*

$$\|\mathbf{u}-\mathbf{u}_h\|_1 + \|p-p_h\|_{L^2(\Omega)} + \|\lambda-\lambda_h\|_{-\frac{1}{2}} \leqslant C(\mathbf{u}, p, g)h^{\frac{3}{4}},$$

where $C(\mathbf{u}, p, g)$ *is a positive constant depending only on* $\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}$, $\|p\|_1$ *and* $\|g\|_{L^2(\Gamma)}$.

5. Numerical test

A no-slip 2D Stokes solver [4] is used and Tresca friction boundary conditions were implemented on. The domain Ω is the square $[0, 0.1]^2$, the fluid can slip on $\Gamma = \Gamma_{upper} \cup \Gamma_{lower} = [0, 0.1] \times \{0.1\} \cup [0, 0.1] \times \{0\}$, the viscosity is taken equal to 0.1 and 10^{-6} is chosen as a stopping criterion. We set $g = 0.015$ which is consistent with experimental values, see [3], and we enforce parabolic profile on both Γ_{left} and Γ_{right}:

$$\mathbf{u}|_{\Gamma_{left}} = \mathbf{u}|_{\Gamma_{right}} = \begin{bmatrix} y(1-y) \\ -y(1-y) \end{bmatrix}.$$

We choose this profile to enforce shear stress near the solid wall to reach the threshold without considering a complicated domain geometry.

Since an explicit solution to such a problem is not available, we calculate the discrete solution with sufficiently refined mesh, $h = \frac{1}{2000}$, which is taken as the reference solution; next we compute $\mathbf{u}_h$, the approximate solution, for different mesh sizes h and we compare them to the reference solution.

Table 1 provides the variation of $\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$, $\|\mathbf{u}-\mathbf{u}_h\|_1$ and $\|p-p_h\|_{L^2(\Omega)}$ with respect to the mesh size respectively. The first remark one can make is the rate convergence of H^1-norm of error on $\mathbf{u}$ is equal to $\frac{3}{4}$ which is in agreement with theoretical result. The second one is that in spite of considering very small mesh size, $h = \frac{1}{1800}$, we cannot conclude about rate convergence of $\mathbf{u}$ and p error L^2-norms; this issue needs to be investigated.

6. Conclusion

A three field mixed formulation of the Stokes problem with Tresca boundary conditions has been introduced and studied. The convergence analysis and a priori error estimates of the discrete corresponding problem have been established. In particular, we show an optimal error estimate of order $h = \frac{3}{4}$ for the velocity when it is approximated by classical *P1 bubble* finite element. A numerical realization of a model example has been proposed which confirms the theoretical result.

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