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Shape optimization for the Maxwell equations under weaker regularity of the data

Dérivée par rapport au domaine dans l'équation de Maxwell sous des hypothèses de plus faible régularité des données

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ABSTRACT

We consider a shape optimization problem for Maxwell's equations with a strictly dissipative boundary condition. In order to characterize the shape derivative as a solution to a boundary value problem, sharp regularity of the boundary traces is critical. This Note establishes the Fréchet differentiability of a shape functional.

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RÉSUMÉ

On considère un problème d'optimisation de forme dans le cadre des équations de Maxwell avec une condition de bord dissipative. On établit un résultat de dérivabilité par rapport au domaine dans le cas de faible régularité. Au détour de cette preuve, on établit la régularité « cachée » des traces du champ éléctrique et magnétique sur le bord du domaine.

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Version française abrégée

Soit $\Omega \subset \mathbb{R}^3$ un ouvert borné connexe de frontière lipschitzienne $\Gamma = \partial \Omega$. L'évolution du champ électrique e = e(t, x) et du champ magnétique h = h(t, x) dans le cylindre d'espace-temps $Q = (0, T) \times \Omega$ est modélisé par les équations de Maxwell (1). La permittivité électrique $\varepsilon = \varepsilon(t, x)$ et la perméabilité magnétique $\mu = \mu(t, x)$ sont des matrices hermitiennes 3×3 définies positives. Les fonctions f_1, f_2 représentent les densités de courant, $\alpha = \alpha(t, x)$ est une fonction positive qui est l'inverse de la conductivité surfacique sur Γ , le rotationel est noté $\nabla \times$ et le temps final est noté T, celui-ci pouvant être infini. La normale unitaire extérieure à Γ est notée ν et on note $h_{\tau} = (\nu \times h) \times \nu = h - (h \cdot \nu)\nu$ la composante tangentielle d'un vecteur h. On note div $_{\Gamma}$ la divergence tangentielle et curl $_{\Gamma}$ le rotationnel tangentiel.

On ajoute les conditions de bord (2), qui sont absorbantes, stricement dissipative, et la condition initiale (3). Le système de Maxwell (1) est symétrique et hyperbolique, il est strictement hyperbolique dans le cas isotrope (c'est-à-dire lorsqu'il existe un réel κ tel que $\varepsilon = \kappa \mu$). Le système peut être ré-écrit sous la forme (4)-(5).

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On introduit $\mathcal{H}(X)$, l'espace des (e, h) dans $L_2(X)^6$ pour lesquels εe et μh sont à divergence nulle. On note A^H la transoposée hermitienne d'une matrice A et A > 0 lorsque celle-ci est définie positive. On considère (E, H) l'état *cible* et $(e - E)^H \varepsilon (e - E) = \varepsilon^{jk} (e - E)_j (\bar{e} - \bar{E})_k$ où $(e, h) = (e, h)(\Omega)$ est solution de (1)-(3).

Dans cette Note on s'interesse à la minimisation de la fonctionnelle (6), le paramêtre d'optimisation est l'ouvert $\Omega \subset \mathbb{R}^3$ sous-ensemble d'un ouvert borné et connexe $D \subset \mathbb{R}^3$.

On se réfère à [6] en ce qui concerne l'introduction aux éléments de dérivée par rapport au domaine. Pour des raisons de simplicité, on présente dans cette note, les resultats dans le cas homogène, avec des coefficients constants et des conditions initiales de divergence nulle.

Théorème 1 (Dérivabilité par rapport au domaine). Soit ε , μ des matrices constantes définies postives, α une constante positive, $f \equiv 0$, $g \equiv 0$ et $(e^0, h^0) \in H^1(D)^6 \cap \mathcal{H}(D)$. On considère

$$(E, H) \in H^1((0, T) \times D) \cap \mathcal{H}((0, T) \times D)$$

Alors, la dérivée eulérienne de la solution de (6) est Fréchet-dérivable en $\Omega \in \mathcal{O}$ dans la direction du champ de vecteur V. La dérivée est donnée par (7) où $V_{\nu} = V(0) \cdot \nu$, $e_{\nu} = e \cdot \nu$, $h_{\nu} = h \cdot \nu$ et (p, q) est la solution au problème adjoint (8).

Pour démontrer ce théorème, on doit établir que la solution du problème direct satisfait $(e, h) \in H^1(Q)^6$, $(e, h)|_{\Sigma} \in H^1(\Sigma)^6$, $(\partial_{\nu} e, \partial_{\nu} h) \in L_2(\Sigma)^6$ et que la solution du problème adjoint satisfait $q|_{\Sigma} \in H^1(\Sigma)^3$. Cela est l'objet du Théorème 3. Nous commençons par établir le

Théorème 2 (Solutions faibles). Supposons Γ de classe C^2 et ε , $\mu \in C^1(\overline{Q})$. Soit $\alpha \in L_\infty(\Sigma)$ tel que $\alpha \ge c > 0$ p.p. Σ . Soit $g \in L_2(\Sigma)^3$ avec $\nu \cdot g = 0$. Soit $f \in L_2(Q)^6$ et la condition initiale $(e^0, h^0) \in L_2(\Omega)^6$. Si $\rho_1, \rho_2 \in L_2(Q)$, alors, il existe une unique solution $(e, h) \in C([0, T], L_2(\Omega)^6)$ à (1)-(3) telle que $(e, h)|_{\Sigma} \in L_2(\Sigma)^6$. En outre, il existe une constante positive γ_0 telle que

$$\|e^{-\gamma T}(e,h)(T)\|_{\Omega}^{2} + \gamma \|e^{-\gamma t}(e,h)\|_{Q}^{2} + \|e^{-\gamma t}(e,h)\|_{\Sigma}^{2} \lesssim \|(e^{0},h^{0})\|_{\Omega}^{2} + \frac{1}{\gamma} \|e^{-\gamma t}(f,\rho)\|_{Q}^{2} + \|e^{-\gamma t}g\|_{\Sigma}^{2}$$

pour $\gamma \ge \gamma_0$.

Les résultats concernant les solutions fortes pour un un système strictement hyperbolique symétrique avec un bord caractéristique peuvent être trouvées dans [5, Theorem 3]. Dans cette référence, il est mise en évidence que le passage des solutions faibles à une solution forte est non-trivial en présence d'un bord caractétistique.

Nous montrons qu'une régularité supplémentaire sur les fonctions ρ_1 et ρ_2 permet d'obtenir une solution forte. Pour y parvenir, nous avons besoins des normes H^1 à poids données dans (12). Nous avons alors le

Théorème 3 (Solutions fortes – Régularité cachée). Supposons Γ de classe C^2 , supposons ε , $\mu \in C^1(\overline{Q})$ et $\alpha \in C^1(\overline{\Sigma})$. Considérons (1)–(3) avec $g \in H^1(\Sigma)^3$ et $\nu \cdot g = 0$ sur Σ , ainsi que $f \in H^1(Q)^6$, $(e^0, h^0) \in H^1(\Omega)^6$ avec la condition de compatibilité $g(0) = \nu \times e^0 - \alpha(0)h_{\tau}^0$ dans $L_2(\Omega)$. Si $\rho_1, \rho_2 \in H^1(Q)$, alors il existe une unique solution forte à (1)–(3) avec

$$(e,h) \in C\left([0,T], H^{1}(\Omega)^{6}\right) \cap C^{1}\left([0,T], L_{2}(\Omega)^{6}\right)$$
$$(e,h)|_{\Sigma} \in H^{1}(\Sigma)^{6}, \qquad (\partial_{\nu}e, \partial_{\nu}h)|_{\Sigma} \in L_{2}(\Sigma)^{6}$$

En outre, il existe une constante γ_0 positive satisfaisant (13) pour $\gamma \ge \gamma_0$.

On remarquera que la regularité de bord obtenue dans les Theorèmes 2 et 3 ne résultent pas du théorème de trace appliqueé à la régularité intérieure. Pour cette raison, ces résultats de régularité entrent dans le cadre de ce qu'on appelle la «régularité cachée ».

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, and connected set with a Lipschitz boundary $\Gamma = \partial \Omega$. The evolution of the electric field e = e(t, x) and the magnetic field h = h(t, x) in the space-time cylinder $Q = (0, T) \times \Omega$ is given by Maxwell's equations

$$\begin{cases} \partial_t (\varepsilon e) - \nabla \times h = f_1 \\ \partial_t (\mu h) + \nabla \times e = f_2 \end{cases} \quad \text{in } Q = (0, T) \times \Omega \tag{1}$$

Here $\varepsilon = \varepsilon(t, x)$ is the electric permittivity, $\mu = \mu(t, x)$ is the magnetic permeability which are both positive definite Hermitian 3×3 matrices. The functions f_1, f_2 represent current densities, $\nabla \times$ denotes the curl operator, ∂_t is the differentiation with respect to t, and the final time T can be infinity. We add the boundary condition

$$\nu \times e - \alpha h_{\tau} = g \quad \text{in } \Sigma = (0, T) \times \partial \Omega \tag{2}$$

and the initial conditions

$$e|_{t=0} = e^0$$
 and $h|_{t=0} = h^0$ in Ω (3)

where ν is the exterior unit normal vector along Γ , $\nu \times e$ is the cross product of the vectors ν and e, $\alpha = \alpha(t, x)$ is a positive function, and $h_{\tau} = (\nu \times h) \times \nu = h - (h \cdot \nu)\nu$ is the tangential component of the vector h on Γ . The boundary condition (3) is an *absorbing boundary* condition. If $\alpha \equiv 1$, this is the Silver–Müller boundary condition. The quantity α represents the inverse of the surface conductivity on Γ and the function g is an external surface current density.

Maxwell's system (1) is known to be symmetric hyperbolic. It can be written

$$\partial_t (A^0 w) + A^j \partial_j w = f \tag{4}$$

$$A^{0} = \begin{bmatrix} \varepsilon & 0\\ 0 & \mu \end{bmatrix}, \qquad A^{j} \partial_{j} = \begin{bmatrix} 0 & -\nabla \times\\ \nabla \times & 0 \end{bmatrix}, \qquad w = \begin{bmatrix} e\\ h \end{bmatrix}, \qquad f = \begin{bmatrix} f_{1}\\ f_{2} \end{bmatrix}$$
(5)

where the summation convention is used. The boundary condition (3) is an example of a *strictly dissipative* boundary condition. Note that the boundary Σ is characteristic, i.e. $\det(A^j v_j) = 0$. Maxwell's equation are strictly hyperbolic only in the isotropic case, that is if $\varepsilon = \kappa \mu$ for some scalar function κ . In the sequel the linear space of Lebesgue measurable functions on the open set *X* with absolutely integrable power *p* is denoted by $L_p(X)$. We write $L_p(X)^N$ for the linear space of vector valued functions with *N* components with each component in $L_p(X)$. For the scalar product in $L_2(X)^N$ we will use the notation $(\cdot, \cdot)_X$, i.e.

$$(w_1, w_2)_X = \int\limits_X w_1 \cdot \bar{w}_2 \,\mathrm{d}X$$

and the corresponding norm is $||w||_X = \sqrt{(w, w)_X}$. The L_2 -based Sobolev spaces will be denoted by $H^s(X)$ for $s \in \mathbf{R}$ and the linear space of k times continuously differentiable functions on the closure of X is denoted by $C^k(\bar{X})$. By $C^k([0, T], Y)$ we denote the linear space of k times continuously differentiable functions with values in the linear space Y. Furthermore,

$$\mathcal{H}(X) = \left\{ (e,h) \in L_2(X)^6 \colon \nabla \cdot (\varepsilon e) = \nabla \cdot (\mu h) = 0 \right\}$$

where $\nabla \cdot (\varepsilon e) = \partial_j (\varepsilon^{jk} e_k)$. If $u : \overline{\Omega} \to \mathbb{R}^3$ is a differentiable function, then $u_v = u \cdot v$ on Γ and $\partial_v u$ is the normal derivative of u along Γ . If A is a Hermitian positive definite matrix we write A > 0. The transpose of a $m \times n$ matrix A is A^T and the Hermitian transpose A^H .

Standard theory of hyperbolic systems provides the following existence, uniqueness and regularity of weak solutions:

Proposition 1. Let ε , μ , $\partial_t \varepsilon$, $\partial_t \mu \in L_{\infty}(Q)$, $\alpha \in L_{\infty}(\Sigma)$ such that ε , $\mu \ge c > 0$ almost everywhere on Q and $\alpha \ge c > 0$ almost everywhere on Σ . Given $f \in L_2(Q)^6$, $g \in L_2(\Sigma)^3$ with $\nu \cdot g = 0$ and $(e^0, h^0) \in L_2(\Omega)^6$ there exists a unique weak solution $(e, h) \in C([0, T], L_2(\Omega)^6)$ such that $(e_{\tau}, h_{\tau})|_{\Sigma} \in L_2(\Sigma)^6$. Moreover, there exists a constant γ_0 such that for all $\gamma \ge \gamma_0$

$$\|e^{-\gamma t}(e,h)(t)\|_{\Omega}^{2} + \gamma \|e^{-\gamma t}(e,h)\|_{Q}^{2} + \|e^{-\gamma t}(e_{\tau},h_{\tau})\|_{\Sigma}^{2} \approx \|(e^{0},h^{0})\|_{\Omega}^{2} + \frac{1}{\gamma} \|e^{-\gamma t}f\|_{Q}^{2} + \|e^{-\gamma t}g\|_{\Sigma}^{2}$$

In the isotropic case, this proposition is a corollary of Theorem 1.12 [5]. In the case of time-independent coefficients, this result can be found in the book by Lagnese and Leugering [3, Chapter 7]. Note that there is no regularity for the normal components of the vectors e and h. This is due to the fact that the boundary Σ is characteristic.

This note is concerned with minimizing the functional

$$J(\Omega) = \frac{1}{2} \int_{Q} \left[(e - E)^{H} \varepsilon (e - E) + (h - H)^{H} \mu (h - H) \right] dt dx$$
(6)

over a collection of bounded, open and connected sets $\Omega \subset \mathbb{R}^3$ with C^2 boundary that are subsets of a fixed bounded, open and connected set $D \subset \mathbb{R}^3$. Here $(e, h) = (e, h)(\Omega)$ is a solution to the initial-boundary value problem (1)-(3), (E, H) is a desired target state and $(e - E)^H \varepsilon (e - E) = \varepsilon^{jk} (e - E)_j (\bar{e} - \bar{E})_k$.

2. Shape optimization problem

In order to discuss the derivative of this functional with respect to Ω , perturbations of the set Ω in direction of a vector field *V* are introduced [6]. Let \mathcal{O} be the collections of open, connected sets $\Omega \subset D$ whose boundary $\Gamma = \partial \Omega$ is of class C^2 . For some small positive number S > 0 we consider a vector field $V \in C([0, S]; C^2(D)^3)$ which is tangential on the boundary ∂D , i.e. $V \cdot v_{\partial D} = 0$ for all $x \in \partial D$. The flow $F_s(x)$ of this vector field is the solution of the ODE

$$\frac{\partial F_s(x)}{\partial s} = V(s, F_s(x)).$$

For *S* sufficiently small this differential equation augmented by the initial condition $F_0(x) = x$ for all $x \in D$ has a unique solution $F_s(x)$ for all $s \in [0, S]$. Furthermore, the solution is of class $C^1([0, S]; C^2(D)^3)$ and is one-to-one and onto. The inverse mapping is also of class $C^1([0, S]; C^2(D)^3)$ [6, Theorem 2.16].

Given $\Omega \in \mathcal{O}$ we obtain a family of perturbed domains $\Omega_s(V) = \{F_s(x): x \in \Omega\}$ which are also in \mathcal{O} .

Now we can state our result regarding the differentiability of the shape functional above. For simplicity we focus on the case with constant coefficients, vanishing right-hand sides and divergence free initial data.

Theorem 1. Assume that ε , μ are constant, Hermitian positive definite matrices and let α be a positive constant. Furthermore, let $f \equiv 0$, $g \equiv 0$ and $(e^0, h^0) \in H^1(D)^6 \cap \mathcal{H}(D)$ and $(E, H) \in H^1((0, T) \times D) \cap \mathcal{H}((0, T) \times D)$. Then, the shape functional is Fréchet differentiable at $\Omega \in \mathcal{O}$ in direction of the vector field V with Fréchet derivative

$$dJ(\Omega, V) = \Re \int_{\Sigma} \left[q \cdot \left\{ \partial_{\nu} \bar{e} \times \nu + \alpha [\partial_{\nu} \bar{h}]_{\tau} \right\} + \operatorname{curl}_{\Gamma} (\bar{e}_{\nu} q) - \alpha \operatorname{div}_{\Gamma} (\bar{h}_{\nu} q) \right] V_{\nu} dt d\Gamma + \frac{1}{2} \int_{\Sigma} \left[(e - E)^{H} \varepsilon (e - E) + (h - H)^{H} \mu (h - H) \right] V_{\nu} dt d\Gamma$$
(7)

where $V_{\nu} = V(0) \cdot \nu$, div_{Γ} is the surface divergence, curl_{Γ} is the surface curl, and (p,q) is the solution to the backward adjoint initial-boundary value problem

$$\begin{cases} \varepsilon \partial_t p - \nabla \times q = \varepsilon (e - E), & \mu \partial_t q + \nabla \times q = \mu (h - H) & \text{in } Q \\ p|_{t=T} = 0, & q|_{t=T} = 0 & \text{in } \Omega \\ \nu \times p + \alpha q_\tau = 0 & \text{in } \Sigma. \end{cases}$$
(8)

This type of results are useful for applications, see e.g. [4] that pertains to the shape optimization of piezoelectric system. The proof of Theorem 1 has two components. One has to establish the shape derivative and establish trace sharp regularity results for weak and differentiable solutions to the initial-boundary value problem under consideration. Those regularity results will guarantee that the expression in (7) is well defined. The function

$$G = \Re q \cdot \left\{ \partial_{\nu} \bar{e} \times \nu + \alpha [\partial_{\nu} \bar{h}]_{\tau} \right\} + \operatorname{curl}_{\Gamma} (\bar{e}_{\nu} q) - \alpha \operatorname{div}_{\Gamma} (\bar{h}_{\nu} q) + \frac{1}{2} (e - E)^{H} \varepsilon (e - E) + \frac{1}{2} (h - H)^{H} \mu (h - H)$$

is the shape gradient of $J(\Omega)$ in direction of V. We have $G \in L_1(\Sigma)$.

The initial-boundary value problem (1)-(3) is weakly shape differentiable and its shape derivative $(e', h') \in C([0, T], L_2(\Omega)^6)$ is a weak solution to the initial-boundary value problem

$$\varepsilon \partial_t e' - \nabla \times h' = \mu \partial_t h' + \nabla \times e' = 0 \quad \text{in } Q \tag{9}$$

with zero initial data

$$u'|_{t=0} = 0 \quad \text{in } \Omega \tag{10}$$

and the lateral boundary condition

$$\nu \times e' - \alpha h'_{\tau} = (V_{\nu} \partial_{\nu} e_{\tau} + e_{\nu} \nabla_{\tau} V_{\nu}) \times \nu + \alpha (V_{\nu} \partial_{\nu} h_{\tau} + h_{\nu} \nabla_{\tau} V_{\nu}) \quad \text{in } \Sigma.$$
⁽¹¹⁾

Proposition 1 is used to characterize the shape derivative as a weak solution to an initial-boundary value problem of the form (1)-(3). Hence one needs to guarantee that the right-hand side in (11) is a tangential vector field in $L_2(\Sigma)$. This is done by showing that the initial-boundary value problem (1)-(3) with data as in Theorem 1 has a unique solution which satisfies

$$(e,h)|_{\Sigma} \in H^1(\Sigma)^6, \qquad (\partial_{\nu} e, \partial_{\nu} h)|_{\Sigma} \in L_2(\Sigma)^6,$$

see Theorem 3 below. This situation is similar to the shape derivative of the Dirichlet problem for the wave equation, studied earlier by Cagnol and Zolésio [1]. We like to point out that our analysis is valid for the anisotropic Maxwell equations which cannot be reduced to vector wave equations.

From the viewpoint of applications the case $\alpha \equiv 0$ may be the most interesting case. It models the boundary Γ as a prefect conductor. In this case the trace regularity results will not hold anymore. However, one can show that the shape functional remains differentiable and that its Fréchet derivative is given by (7) with $\alpha = 0$ where the integral has to interpreted as a dual pairing in $(H^{1/2}(\Sigma), H^{-1/2}(\Sigma))$. This fact complements a recent paper by Jean-Paul Zolésio where shape

differentiability for a shape functional different from (6) in the case of the isotropic Maxwell system is established [7]. We believe that our approach will be useful for boundary value problems with a boundary condition for which the Kreiss–Sakamoto condition (uniform Lopatinskii condition) is not satisfied.

Note that the boundary condition of the shape derivative in the case $\alpha \equiv 0$ simplifies to

$$v \times e' = (V_{v} \partial_{v} e_{\tau} + e_{v} \nabla_{\tau} V_{v}) \times v \text{ in } \Sigma.$$

In the case that ε and μ are scalar functions, it has been shown that the trace of the normal derivative of the electric field is square integrable, i.e. $\partial_{\nu} e|_{\Sigma} \in L_2(\Sigma)^3$ [2, Theorem 1.1], [7, Proposition 6.1]. However, this result is not sufficient to establish the regularity of the shape derivative from the boundary value problem, since Proposition 1 is valid only for positive α .

3. Sharp trace regularity

There is a wealth of results regarding the solutions to boundary value problems for Maxwell's system. However, there are only few results which discuss a non-homogeneous boundary condition and establish regularity of boundary traces. One can obtain regularity results for the normal components of e and h on Σ under additional assumptions. Define the charge densities

$$\rho_1(t,x) = \int_0^t \nabla \cdot f_1(s,x) \,\mathrm{d}s + \nabla \cdot \left(\varepsilon(0,x)e^0(x)\right), \qquad \rho_2(t,x) = \int_0^t \nabla \cdot f_2(s,x) \,\mathrm{d}s + \nabla \cdot \left(\mu(0,x)h^0(x)\right).$$

Theorem 2. Assume that Γ is of class C^2 and let ε , $\mu \in C^1(\overline{Q})$ and let $\alpha \in L_{\infty}(\Sigma)$ such that $\alpha \ge c > 0$ almost everywhere on Σ . Furthermore, let $g \in L_2(\Sigma)^3$ with $\nu \cdot g = 0$, $f \in L_2(Q)^6$, $(e^0, h^0) \in L_2(\Omega)^6$.

If $\rho_1, \rho_2 \in L_2(Q)$, there exists a unique weak solution $(e, h) \in C([0, T], L_2(\Omega)^6)$ to (1)–(3) such that $(e, h)|_{\Sigma} \in L_2(\Sigma)^6$. Moreover, there exists a positive constant γ_0 such that

$$\|e^{-\gamma T}(e,h)(T)\|_{\Omega}^{2} + \gamma \|e^{-\gamma t}(e,h)\|_{Q}^{2} + \|e^{-\gamma t}(e,h)\|_{\Sigma}^{2} \lesssim \|(e^{0},h^{0})\|_{\Omega}^{2} + \frac{1}{\gamma} \|e^{-\gamma t}(f,\rho)\|_{Q}^{2} + \|e^{-\gamma t}g\|_{\Sigma}^{2}$$

for $\gamma \ge \gamma_0$.

The next result is concerned with differentiable solutions to the initial-boundary value problem (1)-(3). To our best knowledge, the only result concerning differentiable solutions for a symmetric hyperbolic system with characteristic boundary can be found in [5, Theorem 3]. This result is only valid for the strictly hyperbolic case. Nevertheless, Theorem 3 in [5] shows that the step from the weak solution to a differentiable solution is a non-trivial matter in the presence of a characteristic boundary.

We will show that additional regularity assumptions on the functions ρ_1 , ρ_2 will produce a differentiable solution. To formulate the related estimate, we need to introduce some weighted H^1 -norms. Define

$$\|u\|_{1,\gamma,Q}^{2} = \gamma^{2} \|e^{-\gamma t}u\|_{Q}^{2} + \|e^{-\gamma t}\partial_{t}u\|_{Q}^{2} + \|e^{-\gamma t}\nabla u\|_{Q}^{2}$$

$$\|u(t)\|_{1,\gamma,\Omega}^{2} = \gamma^{2} \|u(t)\|_{\Omega}^{2} + \|e^{-\gamma t}\nabla u(t)\|_{\Omega}^{2}$$

$$\|u\|_{1,\gamma,\Sigma}^{2} = \gamma^{2} \|e^{-\gamma t}u\|_{\Sigma}^{2} + \|e^{-\gamma t}\partial_{t}u\|_{\Sigma}^{2} + \|e^{-\gamma t}\nabla_{\tau}u\|_{\Sigma}^{2}$$

(12)

for $\gamma > 0$. Here $\nabla u = (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u)$ is the gradient of u and $\nabla_{\tau} u_j = (v \times \nabla u_j) \times v$ is the tangential gradient of u_j along Γ .

Theorem 3. Suppose that Γ is of class C^2 and let ε , $\mu \in C^1(\bar{Q})$ and $\alpha \in C^1(\bar{\Sigma})$. Consider the initial-boundary value problem (1)-(3) with $g \in H^1(\Sigma)^3$ with $v \cdot g = 0$ on Σ , $f \in H^1(Q)^6$, $(e^0, h^0) \in H^1(\Omega)^6$ subject to the compatibility condition $g(0) = v \times e^0 - \alpha(0)h_{\tau}^0$ in $L_2(\Omega)$. If $\rho_1, \rho_2 \in H^1(Q)$, there exists a unique differentiable solution

$$(e, h) \in C([0, T], H^1(\Omega)^6) \cap C^1([0, T], L_2(\Omega)^6)$$

such that $(e,h)|_{\Sigma} \in H^1(\Sigma)^6$ and $(\partial_{\nu} e, \partial_{\nu} h)|_{\Sigma} \in L_2(\Sigma)^6$. Furthermore, there exists a positive constant γ_0 such that

$$\| (e,h)(T) \|_{1,\gamma,\Omega}^{2} + \gamma \| (e,h) \|_{1,\gamma,Q}^{2} + \| (e,h) \|_{1,\gamma,\Sigma}^{2} + \| e^{-\gamma t} (\partial_{\nu} e, \partial_{\nu} h) \|_{\Sigma}^{2}$$

$$\widetilde{\leq} \| (e^{0},h^{0}) \|_{1,\gamma,Q}^{2} + \frac{1}{\gamma} \| (f,\rho) \|_{1,\gamma,Q}^{2} + \| g \|_{1,\gamma,\Sigma}^{2}$$

$$(13)$$

for $\gamma \ge \gamma_0$.

Note that the boundary regularity results obtained in Proposition 1 and Theorems 2 and 3 cannot be inferred from the interior regularity of the solutions and the trace theorem in Sobolev spaces. Hence, these boundary regularity results for hyperbolic problems are referred to as "hidden regularity".

References

- [1] John Cagnol, Jean-Paul Zolésio, Shape derivative in the wave equation with Dirichlet boundary conditions, J. Differential Equations 158 (2) (1999) 175–210.
- [2] Matthias Eller, On boundary regularity of solutions to Maxwell's equations with a homogeneous conservative boundary condition, Discrete Contin. Dyn. Syst. Ser. S 2 (3) (2009) 473-481.
- [3] John E. Lagnese, Günter Leugering, Domain Decomposition Methods in Optimal Control of Partial Differential Equations, Internat. Ser. Numer. Math., vol. 148, Birkhäuser Verlag, Basel, 2004.
- [4] Guenther Leugering, Antonio André Novotny, Gustavo Perla Menzala, Jan Sokołowski, Shape Sensitivity Analysis of a Quasi-Electrostatic Piezoelectric System in Multilayered Media, Math. Methods Appl. Sci., 2010.
- [5] Andrew Majda, Stanley Osher, Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary, Comm. Pure Appl. Math. 28 (5) (1975) 607–675.
- [6] Jan Sokołowski, Jean-Paul Zolésio, Introduction to Shape Optimization: Shape Sensitivity Analysis, Springer Ser. Comput. Math., vol. 16, Springer-Verlag, Berlin, 1992.
- [7] Jean-Paul Zolésio, Hidden boundary shape derivative for the solution to Maxwell equations and non cylindrical wave equations, in: Optimal Control of Coupled Systems of Partial Differential Equations, in: Internat. Ser. Numer. Math., vol. 158, Birkhäuser, Basel, 2009, pp. 319–345.