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Mathematical Analysis

Weighted Paley–Wiener theorem on the Hilbert transform ☆

Version avec poids du théorème de Paley–Wiener sur la transformée de Hilbert

Elijah Liflyand [a], Sergey Tikhonov [b]

[a] *Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel*
[b] *ICREA and Centre de Recerca Matemàtica (CRM), 08193 Bellaterra, Barcelona, Spain*

A R T I C L E   I N F O

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A B S T R A C T

We prove weighted analogues of the Paley–Wiener theorem on integrability of the Hilbert transform of an integrable odd function which is monotone on $\mathbb{R}_+$. This extends Hardy–Littlewood's and Flett's results to the case $p = 1$ under the assumption of (general) monotonicity for an even/odd function.

R É S U M É

Nous prouvons des analogues avec poids du théorème de Paley–Wiener, à savoir l'intégrabilité de la transformée de Hilbert d'une fonction intégrable impaire décroissante sur $\mathbb{R}^+$. Nos résultats étendent au cas $p = 1$ ceux de Hardy–Littlewood et de Flett concernant l'intégrabilité avec poids de la transformée de Hilbert d'une fonction paire ou impaire sous la même condition de décroissance sur $\mathbb{R}^+$ ou sous la condition moins restrictive de « monotonie généralisée ».

Version française abrégée

Il est bien connu que pour la transformée de Hilbert $\mathcal{H}g(x) = p.v. \int_{\mathbb{R}} \frac{g(t)}{t-x}\,dt$ pour le poids $w(x) = |x|^\alpha$ avec $-1 < \alpha < p-1$, on a $\|\mathcal{H}g\|_{L^p(w)} \lesssim \|g\|_{L^p(w)}$, $1 < p < \infty$ [5]. Hardy et Littlewood [4] ont montré que, pour les fonctions paires g, l'inégalité est aussi vraie pour $-p-1 < \alpha < p-1$. Par la suite, Flett [2] a montré le même résultat pour les fonctions impaires sous la condition $-1 < \alpha < 2p-1$.

Lorsque $p = 1$, on sait que seules des inégalités de type faible sont vraies pour la transformation de Hilbert. Par ailleurs, le théorème de Paley–Wiener [9] affirme que pour une fonction monotone impaire décroissante sur $\mathbb{R}_+$, $g \in L^1$ on a $\mathcal{H}g \in L^1$. L'objectif de cette Note est de démontrer des résultats analogues dans un théorème de Paley–Wiener avec poids pour des fonctions paires ou impaires.

Un fonction g, localement à variation bornée sur $\mathbb{R}$, et nulle à l'infini, est dite monotone généralisée, ou $g \in GM$, si elle vérifie (3) et (4), où $C > 1$ and $c > 1$ sont indépendants of x.

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E-mail addresses: liflyand@math.biu.ac.il (E. Liflyand), stikhonov@crm.es (S. Tikhonov).

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Définition. On dit qu'une fonction non négative paire $w \in \Omega$, s'il existe $\varepsilon > 0$ telle que $w(t)t^{1-\varepsilon} \uparrow$ et $w(t)t^{\varepsilon-1} \downarrow$ pour tout $t > 0$, où $\uparrow$ and $\downarrow$ signifie presque croissante ou presque décroissante. On rapelle que h est dite presque croissante (respectivement, décroissante) si $h(x) \le Ch(y)$ ou, de manière équivalente, $h(x) \lesssim h(y)$ si $x < y$ (respectivement, si $x > y$).

Théorème 1. *Soit g un fonction impaire intégrable sur $\mathbb{R}$ pour un poids w. Si $g \in GM$ et $w \in \Omega$, alors sa transformée de Hilbert est aussi intégrable pour le même poids, c'est-à-dire que l'on a* (7)*.*

Définition. On dit qu'une fonction paire non négative $w \in \Omega^*$, s'il existe $\varepsilon > 0$ tel que $w(t)t^{2-\varepsilon} \uparrow$ et $w(t)t^{\varepsilon} \downarrow$ pour tout $t > 0$.

Théorème 2. *Soit g une fonction paire intégrable sur $\mathbb{R}$ pour un poids w. Si $g \in GM$ et $w \in \Omega^*$, alors sa transformée de Hilbert est aussi intégrable avec le même poids, c'est-à-dire que l'on a* (7)*.*

Dans le cas particulier $w(x) = |x|^{\alpha}$, si $g \in GM$, la transformée de Hilbert est intégrable pour ce poids si $-1 < \alpha < 1$ pour g paire et si $-2 < \alpha < 0$ pour g impaire. Ceciétend les résultats de Flett et de Hardy–Littlewood au cas $p = 1$.

1. Introduction

It is well known that the Hilbert transform

$$\mathcal{H}g(x) = p.v. \int_{\mathbb{R}} \frac{g(t)}{t-x}\,\mathrm{d}t$$

is bounded on $L^p(w)$, $1 < p < \infty$, if and only if the weight w is from the Muckenhoupt A_p class [5]. In particular, if $w(x) = |x|^{\alpha}$, where $-1 < \alpha < p - 1$, then

$$\|\mathcal{H}g\|_{L^p(w)} \lesssim \|g\|_{L^p(w)}. \tag{1}$$

Hardy and Littlewood [4] showed that for even functions g, inequality (1) with $1 < p < \infty$ also holds for $w(x) = |x|^{\alpha}$, where $-p - 1 < \alpha < p - 1$. Later, Flett [2] proved the same results for odd functions provided $-1 < \alpha < 2p - 1$. Finally, Andersen [1] found complete characterizations of those w satisfying (1) for odd and even functions.

When $p = 1$, it is known that only weak type inequalities for the Hilbert transform hold. In this case, the correct characterization is the Muckenhoupt A_1 class, in particular, $w(x) = |x|^{\alpha} \in A_1$, when $-1 < \alpha \leqslant 0$.

No strong type inequalities (1) with $p = 1$ hold for $w \in A_1$, even under the assumption of oddness or evenness of g (see examples in Section 2 below). On the other hand, Paley–Wiener's theorem [9] asserts that for an odd and monotone decreasing on $\mathbb{R}_+$ function $g \in L^1$ one has $\mathcal{H}g \in L^1$, i.e., g is in the (real) Hardy space $H^1(\mathbb{R})$ (for alternative proof and discussion, see, e.g., Zygmund's paper [11]). The oddness of g is essential, since by Kober's result [6], if $g \in H^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} g(t)\,\mathrm{d}t = 0. \tag{2}$$

The goal of this note is to prove the weighted analogues of the Paley–Wiener theorem for odd and even functions. In particular, we show that for a weight $w(x) = |x|^{\alpha}$, the Hilbert transform is bounded in $L(w)$, when $-1 < \alpha < 1$ provided that g is odd and monotone on $\mathbb{R}_+$ or, when $-2 < \alpha < 0$ provided that g is even and monotone on $\mathbb{R}_+$. Thus assuming monotonicity or general monotonicity of g allows us to extend Flett's and Hardy–Littlewood's results for $p = 1$.

A function g, which is locally of bounded variation on $\mathbb{R}$ and vanishes at infinity, is said to be general monotone (see [7,10]), or $g \in GM$, if it satisfies the conditions

$$\int_{x}^{2x} \left|\mathrm{d}g(t)\right| \leqslant C \int_{x/c}^{cx} \frac{|g(t)|}{t}\,\mathrm{d}t, \quad x \in (0, \infty), \tag{3}$$

and

$$\int_{2x}^{x} \left|\mathrm{d}g(t)\right| \leqslant C \int_{x/c}^{cx} \frac{|g(t)|}{t}\,\mathrm{d}t, \quad x \in (-\infty, 0), \tag{4}$$

where $C > 1$ and $c > 1$ are independent of x. If g is even or odd, then both conditions are the same. Note that any monotone or quasi-monotone function g is general monotone.

Definition. Let a non-negative even function, or weight, w belong to the Ω class, written $w \in \Omega$, if there exists $\varepsilon > 0$ such that

$$w(t)t^{1-\varepsilon} \uparrow \quad \text{for all } t > 0, \tag{5}$$

$$w(t)t^{\varepsilon-1} \downarrow \quad \text{for all } t > 0, \tag{6}$$

where $\uparrow$ and $\downarrow$ mean almost increase and almost decrease.

Recall that h is called almost increasing (respectively, decreasing) if $h(x) \leqslant Ch(y)$ or, equivalently, $h(x) \lesssim h(y)$ when $x < y$ (respectively, $x > y$).

Theorem 1. *Let g be an odd function integrable on $\mathbb{R}$ with a weight w, i.e., $\|g\|_{L(w)} = \int_{\mathbb{R}} |g|w < \infty$. If $g \in GM$ and $w \in \Omega$, then*

$$\|\mathcal{H}g\|_{L(w)} \lesssim \|g\|_{L(w)}. \tag{7}$$

A counterpart for even functions reads as follows:

Definition. Let a weight w belong to the Ω^* class, written $w \in \Omega^*$, if there exists $\varepsilon > 0$ such that

$$w(t)t^{2-\varepsilon} \uparrow \quad \text{for all } t > 0, \tag{8}$$

$$w(t)t^{\varepsilon} \downarrow \quad \text{for all } t > 0. \tag{9}$$

Theorem 2. *Let g be an even function integrable on $\mathbb{R}$ with weight w. If $g \in GM$ and $w \in \Omega^*$, then* (7) *holds.*

Note that $w(t) \in \Omega$ if and only if $|t|w(t) \in \Omega^*$. Also, if $w \in \Omega$, then w is from the A_2 class. Finally, if $w \in \Omega \cap \Omega^*$, then it can be shown that $w \in A_1$. A non-weighted version of Theorem 1 was proved in [8].

Not posing assumptions of evenness or oddness, we obtain a weighted estimate for the Hilbert transform of a function integrable on the whole $\mathbb{R}$.

Corollary 3. *Let $w \in \Omega \cap \Omega^*$, i.e., w satisfy* (5) *and* (9)*. If $g \in GM$, then* (7) *holds.*

In particular, for the weight $w(x) = |x|^{\alpha}$, $-1 < \alpha < 0$, the Hilbert transform $\mathcal{H}g$ is in $L(w)$ provided $g \in L(w) \cap GM$.

2. Proofs

Proof of Theorem 1. Since g is odd and w is even, $\|\mathcal{H}g\|_{L(w)} = 2\int_0^\infty |\mathcal{H}g(u)|w(u)\,\mathrm{d}u$ and

$$\int_0^\infty w(u)\left|\left(\int_{3u/2}^\infty + \int_{-\infty}^{-3u/2}\right)\frac{g(t)}{u-t}\,\mathrm{d}t\right|\mathrm{d}u \leqslant \int_0^\infty w(u)\int_{3u/2}^\infty \left|g(t)\right|\frac{2t}{t^2-u^2}\,\mathrm{d}t\,\mathrm{d}u \lesssim \int_0^\infty t\left|g(t)\right|\int_0^{2t/3}\frac{w(u)}{t^2-u^2}\,\mathrm{d}u\,\mathrm{d}t. \tag{10}$$

Applying (5), we get

$$\int_0^{2t/3}\frac{w(u)}{t^2-u^2}\,\mathrm{d}u \lesssim \frac{w(t)t^{1-\varepsilon}}{t^2}\int_0^{2t/3} u^{\varepsilon-1}\,\mathrm{d}u \lesssim w(t)t^{-1}.$$

By this, the right-hand side of (10) is dominated by $\|g\|_{L(w)}$.

Similarly, but making use of (6), we get

$$\int_0^\infty w(u)\left|\left(\int_0^{u/2} + \int_{-u/2}^0\right)\frac{g(t)}{u-t}\,\mathrm{d}t\right|\mathrm{d}u \lesssim \int_0^\infty t\left|g(t)\right|\int_{2t}^\infty \frac{w(u)}{u^2-t^2}\,\mathrm{d}u\,\mathrm{d}t$$

$$\lesssim \int_0^\infty t\left|g(t)\right|\int_{2t}^\infty \frac{4w(u)u^{\varepsilon-1}}{3u^{2+\varepsilon-1}}\,\mathrm{d}u\,\mathrm{d}t \lesssim \int_0^\infty w(t)\left|g(t)\right|\mathrm{d}t.$$

We remark that (5) and (6) imply

$$w(u) \asymp w(t), \qquad t \in [\beta u, \gamma u],\ 0 < \beta < \gamma, \tag{11}$$

i.e., $C_1 w(u) \leqslant w(t) \leqslant C_2 w(u)$, where $C_1, C_2 > 0$ are independent of t and u. Then

$$\int_0^\infty w(u)\left|\int_{-3u/2}^{-u/2} \frac{g(t)}{u-t}\,\mathrm{d}t\right|\mathrm{d}u \lesssim \int_0^\infty |g(t)| \int_{2t/3}^{2t} \frac{w(u)}{u+t}\,\mathrm{d}u\,\mathrm{d}t \lesssim \int_0^\infty w(t)\big|g(t)\big|\,\mathrm{d}t.$$

Therefore, collecting estimates from above,

$$\int_0^\infty w(u)\left|\int_{-\infty}^{\infty} \frac{g(t)}{u-t}\,\mathrm{d}t\right|\mathrm{d}u \lesssim \int_0^\infty \left|w(u)\int_{u/2}^{3u/2} \frac{g(t)}{u-t}\,\mathrm{d}t\right|\mathrm{d}u + \int_0^\infty w(t)\big|g(t)\big|\,\mathrm{d}t \lesssim I + \int_0^\infty w(t)\big|g(t)\big|\,\mathrm{d}t,$$

where

$$I = \int_0^\infty w(u)\left|\int_0^{u/2} \big[g(u+t) - g(u-t)\big]\frac{\mathrm{d}t}{t}\right|\mathrm{d}u.$$

We then have

$$I \leqslant \int_0^\infty w(u)\int_0^{u/2}\left(\int_{u-t}^{u+t}\big|\mathrm{d}g(s)\big|\right)\frac{\mathrm{d}t}{t}\,\mathrm{d}u \leqslant \int_0^\infty\int_{2t}^\infty w(u)\left(\int_{u-t}^{u+t}\big|\mathrm{d}g(s)\big|\right)\mathrm{d}u\,\frac{\mathrm{d}t}{t}$$

$$= \int_0^\infty\left[\int_t^{3t}\big|\mathrm{d}g(s)\big|\int_{2t}^{s+t} w(u)\,\mathrm{d}u + \int_{3t}^\infty\big|\mathrm{d}g(s)\big|\int_{s-t}^{s+t} w(u)\,\mathrm{d}u\right]\frac{\mathrm{d}t}{t} =: I_1 + I_2.$$

By (11),

$$\frac{1}{t}\int_{2t}^{s+t} w(u)\,\mathrm{d}u \leqslant \frac{1}{t}\int_{2t}^{4t} w(u)\,\mathrm{d}u \asymp w(t), \quad s \in [t, 3t]$$

and

$$\frac{1}{t}\int_{s-t}^{s+t} w(u)\,\mathrm{d}u \asymp w(s), \quad s \geqslant 3t.$$

Hence, since $g \in GM$,

$$I_1 \lesssim \int_0^\infty w(t)\left(\int_t^{3t}\big|\mathrm{d}g(s)\big|\right)\mathrm{d}t \lesssim \int_0^\infty w(t)\left(\int_{t/c}^{ct}\frac{|g(s)|}{s}\,\mathrm{d}s\right)\mathrm{d}t \lesssim \int_0^\infty \big|g(s)\big|\left(\int_{t/c}^{ct}\frac{w(t)}{t}\,\mathrm{d}t\right)\mathrm{d}s \asymp \int_0^\infty \big|g(s)\big|w(s)\,\mathrm{d}s.$$

Changing the order of integration yields

$$I_2 \lesssim \int_0^\infty\left(\int_{3t}^\infty w(s)\big|\mathrm{d}g(s)\big|\right)\mathrm{d}t \lesssim \int_0^\infty s w(s)\big|\mathrm{d}g(s)\big| \lesssim \int_0^\infty\int_t^{2t} s w(s)\big|\mathrm{d}g(s)\big|\frac{\mathrm{d}t}{t}.$$

Using (11) and general monotonicity of g, we get

$$I_2 \lesssim \int_0^\infty w(t)\int_t^{2t}\big|\mathrm{d}g(s)\big|\,\mathrm{d}t \lesssim \int_0^\infty \big|g(s)\big|w(s)\,\mathrm{d}s. \qquad \square$$

Proof of Theorem 2. The proof goes along the same lines as in Theorem 1. Using the evenness of g, we obtain

$$\int_0^\infty w(u)\left|\left(\int_{3u/2}^\infty + \int_{-\infty}^{-3u/2}\right)\frac{g(t)}{u-t}\,\mathrm{d}t\right|\mathrm{d}u \lesssim \int_0^\infty \big|g(t)\big|\int_0^{2t/3}\frac{u w(u)}{t^2-u^2}\,\mathrm{d}u\,\mathrm{d}t \lesssim \int_0^\infty w(t)\big|g(t)\big|\,\mathrm{d}t, \tag{12}$$

since, by (8), we have

$$\int_0^{2t/3} \frac{uw(u)}{t^2-u^2}\,\mathrm{d}u \lesssim \frac{w(t)t^{2-\varepsilon}}{t^2} \int_0^{2t/3} u^{\varepsilon-1}\,\mathrm{d}u \lesssim w(t).$$

Taking into account (9), we get

$$\int_0^\infty w(u)\left|\left(\int_0^{u/2} + \int_{-u/2}^{0}\right)\frac{g(t)}{u-t}\,\mathrm{d}t\right|\mathrm{d}u \lesssim \int_0^\infty \bigl|g(t)\bigr| \int_{2t}^\infty \frac{uw(u)}{u^2-t^2}\,\mathrm{d}u\,\mathrm{d}t \lesssim \int_0^\infty \bigl|g(t)\bigr| \int_{2t}^\infty \frac{w(u)u^\varepsilon}{u^{1+\varepsilon}}\,\mathrm{d}u\,\mathrm{d}t \lesssim \int_0^\infty w(t)\bigl|g(t)\bigr|\,\mathrm{d}t.$$

Finally, we note that (8) and (9) also imply (11) and we can repeat the rest of the proof of Theorem 1. □

Proof of Corollary 3. Representing g in a standard way as the sum of its even and odd parts

$$g(t) = \frac{g(t)+g(-t)}{2} + \frac{g(t)-g(-t)}{2},$$

we apply the same calculations as in the proof of Theorem 1 to the odd part and of Theorem 2 to the even part. Using then $|g(t) \pm g(-t)| \leqslant |g(t)| + |g(-t)|$ and $w \in \Omega \cap \Omega^*$, we obtain the required estimate. □

Examples. There exists an odd function with non-integrable Hilbert transform: take $g(t) = (t-1)^{-1}|\ln^{-2}(t-1)|$ on $(1, 3/2)$, $g(t) = -g(-t)$ on $(-3/2, -1)$, and 0 otherwise. Then for $x \in (1/2, 1)$

$$\bigl|\mathcal{H}g(x)\bigr| \geqslant \left|\int_1^{1+(1-x)} \frac{1}{(t-1)\ln^2(t-1)}\frac{\mathrm{d}t}{t-x}\right| - \frac{2}{3\ln 2} \geqslant \frac{1}{2(1-x)|\ln(1-x)|} - \frac{2}{3\ln 2},$$

which is obviously non-integrable. Similarly, an example in the even case is a modification of Pitt's example given in [6, Theorem 1(b)]: taking $g_1(t) = t^{-1}\ln^{-2} t$ and $g_2(t) = 2(\ln 2)^{-1}$ in $(0, 1/2)$, $g_1(t) = g_2(t) = 0$ otherwise, $g(t) = g_1(t) - g_2(t)$. This function satisfies (2), is integrable on $\mathbb{R}$ and, by routine calculations as above, its Hilbert transform does not belong to $L_1(-1/2, 0)$. It remains to extend it even and take into account that the even extension possesses the same properties (see [3, Lemma 7.40, p. 354]).

3. Periodic functions

Using the same techniques, we can transfer the obtained results to the periodic setting. Recall that for 2π-periodic and integrable function g, its conjugate function is

$$\widetilde{g}(u) = p.v. \int_{-\pi}^{\pi} g(t)\cot\frac{t-u}{2}\,\mathrm{d}t.$$

We use the same notations for an even and π-periodic weight w: $w \in \Omega$ and $w \in \Omega^*$ if conditions (5)–(9), respectively, are satisfied for $0 < t < \pi/2$. Then the periodic analogues of Theorems 1 and 2 are now given as follows:

Theorem 4. *Let g be an odd function integrable on $[-\pi, \pi)$ with a weight w, i.e., $\|g\|_{L^*(w)} = \int_{-\pi}^{\pi} |g|w < \infty$. If $g \in GM$ and $w \in \Omega$, then*

$$\|\widetilde{g}\|_{L^*(w)} \lesssim \|g\|_{L^*(w)}. \tag{13}$$

Theorem 5. *Let g be an even function integrable on $[-\pi, \pi)$ with a weight w. If $g \in GM$ and $w \in \Omega^*$, then* (13) *holds.*

The proofs are similar to the proofs of Theorems 1 and 2. Let us outline the points where certain difference may occur. First of all, we observe that (cf., e.g., [2, Lemma 3])

$$\left|\cot\frac{t-u}{2} + \cot\frac{t+u}{2}\right| = \left|\frac{\sin t}{\sin\frac{t-u}{2}\sin\frac{t+u}{2}}\right| \lesssim \frac{t}{|t^2-u^2|}, \quad 0 < t, u < \pi$$

and

$$\left|\cot\frac{t-u}{2} - \cot\frac{t+u}{2}\right| \lesssim \frac{u}{|t^2-u^2|}, \quad 0 < t, u < \pi.$$

The first bound appears while working with an odd g, the second one fits the case when g is even. Both estimates allow one to work with the same kernel as for the Hilbert transform and thus repeat the same calculations as in the proof of Theorems 1 and 2.

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