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Twisted index theory for foliations

Théorie de l'indice tordue pour des feuilletages

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ABSTRACT

For any Lie groupoid with a twisting, we define an analytic index morphism using the Connes tangent groupoid. This morphism agrees with the one of the Lie groupoid when the twisting is trivial. We discuss a longitudinal index theorem, geometric cycles, push-forward maps and Baum–Connes assembly maps for foliations with a twisting on the space of leaves.

RÉSUMÉ

Nous utilisons le groupoïde tangent de Connes pour définir le morphisme d'indice analytique d'un groupoïde de Lie muni d'un twisting. Lorsque le twisting est trivial, ce morphisme coïncide avec l'indice analytique du groupoïde. Pour le cas de feuilletages, nous montrons un théorème de l'indice longitudinal, nous discutons la construction des morphismes $f_!$ et du morphisme d'assemblage à la Baum–Connes.

Version française abrégée

Définition 0.1. Un twisting α sur un groupoïde de Lie $\mathcal{G} \rightrightarrows M$ est donné par un $PU(H)$-fibré principal sur $\mathcal{G}$. De manière équivalente, un twisting est donné par un morphisme (généralisé) de Hilsum–Skandalis $\alpha : \mathcal{G} \to PU(H)$.

Étant donné un twisting α sur un groupoïde de Lie $\mathcal{G}$, on a des twistings induits naturellement sur M (les unités du groupoïde vu comme groupoïde identité) et sur $A\mathcal{G}$ (l'algébroide de $\mathcal{G}$ vu comme groupoïde en utilisant sa structure de fibré vectoriel). On les note par α_0 et $\pi^*\alpha_0$ respectivement.

Proposition 0.2. *Soit $(\mathcal{G}, \alpha)$ un groupoïde avec un twisting. Il existe un twisting α^T sur le groupoïde tangent de $\mathcal{G}$, $\mathcal{G}^T = A\mathcal{G} \times \{0\} \sqcup \mathcal{G} \times (0,1]$, tel que $\alpha^T|_{A\mathcal{G}} = \pi^*\alpha_0$ et $\alpha^T|_{\mathcal{G}_t} = \alpha$ pour $t \neq 0$.*

Alors, on a une suite exacte de C^*-algèbres [10] $0 \to C^*(\mathcal{G} \times (0,1], p^*_{(0,1]}\alpha) \to C^*(\mathcal{G}^T, \alpha^T) \xrightarrow{ev_0} C^*(A\mathcal{G}, \alpha_0) \to 0$. La C^*-algèbre $C^*(\mathcal{G} \times (0,1], p^*_{(0,1]}\alpha)$ est contractile. Alors si on applique le foncteur de K-théorie à la suite précédente on obtient

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que $K_i(C^*(\mathcal{G}^T,\alpha^T)) \xrightarrow{(ev_0)_*} K_i(C^*(A\mathcal{G},\alpha_0))$ est un isomorphisme. Nous définissons $ind^a_{(\mathcal{G},\alpha)} = (ev_1)_* \circ (ev_0)_*^{-1} \circ \mathcal{F}$. Dans le cas où le twisting est trivial on récupère l'indice analytique du groupoïde [7,9].

Proposition 0.3. *Le morphisme d'indice $ind^a_{(\mathcal{G},\alpha)}$ satisfait les trois propriétés suivantes :*

(i) *Compatibilité avec le morphisme de Bott.*
(ii) *Compatibilité avec des sous-groupoïdes ouverts.*
(iii) *Soit $N \to T$ un fibré réel vectoriel et $N \times_T N \rightrightarrows N$ le groupoïde produit associé. Alors l'indice tordu $ind_{a,(N\times_T N,\beta\circ\mathcal{M})}$ coïncide avec l'inverse du morphisme de Thom tordu* [1] *modulo une équivalence de Morita et l'isomorphisme de Fourier.*

Pour le cas des feuilletages réguliers, l'indice topologique à la Connes–Skandalis peut être défini dans le cas tordu (voir définition 2.1 ci-dessous). Le premier résultat important de cette note est la version tordue du théorème longitudinal de Connes–Skandalis.

Théorème 0.4. *De plus, pour un feuilletage régulier (M,F) muni d'un twisting $\alpha : M/F \to PU(H)$, on a l'égalité des morphismes : $ind_{a,(M/F,\alpha)} = ind_{t,(M/F,\alpha)}$.*

Pour un morphisme lisse de Hilsum–Skandalis $f : W \to M/F$ et un twisting α sur M/F, on peut aussi construire le morphisme $f_! : K^{n_W+i}(W, f^*(\alpha_0 + o_\tau) + o_{TW}) \to K^i(M/F,\alpha)$ (définition 3.1). Le résultat principal est :

Théorème 0.5. *La construction $f_!$ est fonctorielle.*

La définition de cycles géométriques tordus introduite dans [12] s'étend au cas des feuilletages en utilisant les morphismes généralisés. On obtient, comme dans [5] pour le cas non tordu, le résultat suivant :

Théorème 0.6. *L'élément $\mu_\alpha(W,f,\eta,[E]) = f_!([E]) \in K^*(M/F,\alpha)$ ne dépend que de la classe d'équivalence du cycle tordu $(W,f,\eta,[E])$ dans $K^{geo}_*(M/F,\alpha + o_{\tau_M})$. En particulier, on a un morphisme d'assemblage $\mu_\alpha : K^{geo}_*(M/F,\alpha+o_{\tau_M}) \to K^*(M/F,\alpha)$.*

1. Twistings on Lie groupoids, twisted *K*-theory and analytic index morphisms

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with the source map s and the range map r, and $PU(H)$ be the projective unitary group with the norm topology for an infinite dimensional, complex and separable Hilbert space H.

Definition 1.1. A twisting α on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is given by a locally trivial right principal $PU(H)$-bundle over $\mathcal{G}$, that is, a locally trivial right principal $PU(H)$-bundle P over M which also admits a left $\mathcal{G}$-action such that the two actions commute. Then $(\mathcal{G},\alpha)$ is called a twisted Lie groupoid.

With respect to an open cover $\Omega = \{\Omega_i\}$ of the units M, a principal $PU(H)$-bundle over $\mathcal{G}$ is defined by a $PU(H)$-valued 1-cocycle $\alpha_{ij} : \mathcal{G}_i^j \to PU(H)$, here $\mathcal{G}_i^j = \{\gamma \in \mathcal{G} | s(\gamma) \in \Omega_i, r(\gamma) \in \Omega_j\}$. Then a twisting α is given by a strict morphism of groupoids from the associated covering groupoid $\mathcal{G}_\Omega = \bigsqcup_{i,j} \mathcal{G}_i^j \rightrightarrows \bigsqcup_i \Omega_i$ to $PU(H) \rightrightarrows \{e\}$. Equivalently, a twisting α is given by a Hilsum–Skandalis morphism $\alpha : \mathcal{G} \to PU(H)$ as defined in [7].

For a twisted Lie groupoid $(\mathcal{G},\alpha)$, there is an induced S^1-central extension over $\mathcal{G}_\Omega$ by taking the pullback of the canonical S^1-central extension $S^1 \to U(H) \to PU(H)$ by the strict morphism $\alpha : \mathcal{G}_\Omega \to PU(H)$. Let us denote it as $S^1 \to R_\alpha \to \mathcal{G}_\Omega$. Take $L_\alpha := R_\alpha \times_{S^1} \mathbb{C}$ the associated complex line bundle over $\mathcal{G}_\Omega$. Using the groupoid structure of R_α, we have a Fell bundle structure on L_α. This leads to a canonical convolution product on $C_c^\infty(\mathcal{G}_\Omega, L_\alpha)$, the space of compactly supported sections of L_α.

Definition 1.2. The twisted groupoid C^*-algebra $C^*(\mathcal{G},\alpha)$ is the C^*-algebra associated to the Fell bundle L_α over $\mathcal{G}_\Omega$, i.e., it is a C^*-completion of the compactly supported sections $C_c^\infty(\mathcal{G}_\Omega, L_\alpha)$, for more details see [10,11]. The twisted K-theory groups of $(\mathcal{G},\alpha)$ are defined as $K^i(\mathcal{G},\alpha) := K_{-i}(C^*(\mathcal{G},\alpha))$, the K-theory groups of the twisted groupoid C^*-algebra $C^*(\mathcal{G},\alpha)$.

We remark that the S^1-action on R_α induces a $\mathbb{Z}$-gradation in $C_r^*(R_\alpha)$ (Proposition 3.2 [11]) which leads to decompositions $C^*(R_\alpha) \cong \bigoplus_{n\in\mathbb{Z}} C^*(\mathcal{G},\alpha^n)$ and $K_i(C^*(R_\alpha)) \cong \bigoplus_{n\in\mathbb{Z}} K_i(C^*(\mathcal{G},\alpha^n))$. The restriction of α to the units M is denoted by α_0 with the corresponding central extension $S^1 \to R_{\alpha_0} \to M_\Omega := \bigsqcup_{i,j} \Omega_{ij} \rightrightarrows \bigsqcup_i \Omega_i$. Then the twisted K-theory groups of the twisted groupoid C^*-algebra $C^*(M_\Omega,\alpha_0)$ agree with the topological twisted K-theory of (M,α_0).

Consider the S^1-equivariant immersion of Lie groupoids $R_{\alpha_0} \overset{\iota_\alpha}{\hookrightarrow} R_\alpha$. Let R^N be the total space of the normal bundle to ι_α. Consider the Lie algebroid of $\mathcal{G}$, as a groupoid $A\mathcal{G} \rightrightarrows M$ (using its vector bundle structure). We use the notation $\pi^*\alpha_0$ for the pull-back twisting of α_0 by the projection $\pi : A\mathcal{G} \to M$ (as a groupoid morphism). A direct computation shows that R^N as a groupoid over the units $\bigsqcup_i \Omega_i$ is the S^1-central extension over $(A\mathscr{G})_\Omega \rightrightarrows \bigsqcup_i \Omega_i$ defined by the twisting $\pi^*\alpha_0 : (A\mathscr{G})_\Omega = \bigsqcup_{i,j} \pi^{-1}(\Omega_{ij}) \to PU(H)$. In particular $C^*(R^N) = \bigoplus_{n\in\mathbb{Z}} C^*(A\mathscr{G}, \alpha_0^n)$.

We apply the tangent groupoid construction to the immersion ι_α to get $R^T_{\iota_\alpha} := R^N \times \{0\} \bigsqcup R_\alpha \times (0,1] \rightrightarrows \bigsqcup_i \Omega_i \times [0,1]$. The exact sequence of C^*-algebras $0 \to C^*(R_\alpha \times (0,1]) \to C^*(R^T_{\iota_\alpha}) \xrightarrow{ev_0} C^*(R^N) \to 0$ and the evaluation map $ev_1 : R^T_{\iota_\alpha} \to R_\alpha$ define an index morphism as in [7], $ind_{\iota_\alpha} = (ev_1)_* \circ (ev_0)_*^{-1} : K_*(C^*(R^N)) \to K_*(C^*(R_\alpha))$. Since the inclusion ι_α is S^1-equivariant, the index map ind_{ι_α} respects the gradation. Therefore, we have a morphism $ind_{\iota_\alpha} : K^i(A\mathscr{G}, \pi^*\alpha_0) \to K^i(\mathscr{G}, \alpha)$.

Definition 1.3. Let $(\mathscr{G}, \alpha)$ be a twisted Lie groupoid. Its **analytic index morphism** is defined to be

$$ind^a_{(\mathscr{G},\alpha)} = ind_{\iota_\alpha} \circ \mathscr{F} : K^i\big(A^*\mathscr{G}, \pi^*\alpha_0\big) \to K^i(\mathscr{G}, \alpha) \tag{1}$$

where $\mathscr{F}$ is the isomorphism, induced by the fiberwise Fourier transform, between the K-theory groups of the twisted groupoid C^*-algebra $C^*(A\mathscr{G}, \pi^*\alpha_0)$ and the topological twisted K-theory of $(A^*\mathscr{G}, \pi^*\alpha_0)$.

There is an alternative way of constructing the analytic index morphism without considering directly the immersion of S^1-central extensions. The proof is an application of the functoriality of the deformation to the normal cone functor established in [2]. Given a twisting α on a Lie groupoid $\mathscr{G}$, there is a twisting α^T on its tangent groupoid $\mathscr{G}^T = A\mathscr{G} \times \{0\} \sqcup \mathscr{G} \times (0,1] \rightrightarrows M \times [0,1]$ such that $\alpha^T|_{A\mathscr{G}} = \pi^*\alpha_0$, and $\alpha^T|_{\mathscr{G}_t} = \alpha$ for $t \neq 0$. Hence we have an exact sequence of C^*-algebras (cf. [10])

$$0 \to C^*\big(\mathscr{G} \times (0,1], \alpha\big) \to C^*\big(\mathscr{G}^T, \alpha^T\big) \xrightarrow{Ev_0} C^*(A\mathscr{G}, \alpha_0) \to 0.$$

This, together with $Ev_1 : C^*(\mathscr{G}^T, \alpha^T) \to C^*(\mathscr{G}, \alpha)$, gives rise to an equivalent definition of the above analytic index morphism $ind^a_{(\mathscr{G},\alpha)} = (Ev_1)_* \circ (Ev_0)_*^{-1} \circ \mathscr{F}$. When the twisting α is trivial, we recover the analytic index morphism associated to any Lie groupoid [7,9].

Now, the analytic index morphism we just defined satisfies the following axioms:

Proposition 1.4. *The analytic index morphism in Definition* 1.3 *satisfies the three following properties*:

(i) *It is compatible with the Bott morphism*: $B \circ ind^a_{(\mathscr{G},\alpha)} = ind^a_{(\mathscr{G}\times\mathbb{R}^2, \alpha\circ p)} \circ B$, *where B stands for the classical Bott morphism of C^*-algebras, and $p : \mathscr{G} \times \mathbb{R}^2 \to \mathscr{G}$ denotes the projection.*

(ii) *It is compatible with open inclusions*: *Let $\mathscr{H} \overset{j}{\hookrightarrow} \mathscr{G}$ be an open inclusion of groupoids and dj be the induced inclusion of their Lie algebroids, then $j_! \circ ind^a_{(\mathscr{H},\alpha\circ j)} = ind^a_{(\mathscr{G},\alpha)} \circ (dj)_!$.*

(iii) *It is compatible with the topological Thom isomorphism*: *Let $N \to T$ be a real vector bundle. Consider the groupoid $\mathscr{N} := N \times_T N \rightrightarrows N$ associated to the submersion $N \to T$, which is Morita equivalent to the identity groupoid $T \rightrightarrows T$. Let β be a twisting on the space T, then the inverse of the topological Thom isomorphism $K_i(N \oplus N, \pi_T^*\beta) \cong K_i(T, \beta)$ in* [1] *agrees with the composition of the analytic index $ind^a_{(N\times_T N, \beta\circ\mathscr{M})}$ and the isomorphism from the Morita equivalence $\mathscr{M} : \mathscr{N} \to T$.*

A note on the proof. For (i) it is worth to remark that once we consider the associated central extensions, the Bott periodicity is given by the usual Bott morphism in C^*-algebras. And it is immediate that it commutes with evaluation morphism by naturality of the product in K-theory. The morphisms $j_!$ and $(dj)_!$ in (ii), at the level of extensions, come from morphisms of algebras and hence commute with the evaluations. For (iii), when the twisting is trivial it reduces to Theorem 6.2 in [6]. In the twisted case, we have locally the situation of Debord–Lescure–Nistor, we have then to use the transition data given by the corresponding Fell bundle L_α to obtain our result by Mayer–Vietoris arguments.

2. Longitudinal twisted index theorem for foliations

Consider the case of a regular foliation (M, F) with a twisting α on its leaf space M/F, where M/F is identified with the holonomy groupoid $\mathscr{G}$ of (M, F). The induced twisting on M is denoted by α_0 as in Section 1. Note that the Lie algebroid $A\mathscr{G}$ is $F \rightrightarrows M$, as a vector bundle groupoid. We adapt the definition of the topological index of Connes–Skandalis [5] to $(M/F, \alpha)$.

Let $i : M \hookrightarrow \mathbb{R}^{2m}$ be an embedding and T be the total space of the normal vector bundle $\pi_T : T \to M$ to the foliation in $\mathbb{R}^{2m}$: $T_x = (i_*(F_x))^\perp$. Consider the foliation $M \times \mathbb{R}^{2m}$ given by $\tilde{F} = F \times \{0\}$. This foliation has $\tilde{\mathscr{G}} = \mathscr{G} \times \mathbb{R}^{2m}$ as its holonomy groupoid equipped with the pull-back twisting, still denoted by α. The map $(x, \xi) \mapsto (x, i(x) + \xi)$ identifies an open neighborhood of the 0-section of T with an open transversal of $(M \times \mathbb{R}^{2m}, \tilde{F})$, that we still denote by T. Let N be the normal vector bundle to the inclusion $T \subset M \times \mathbb{R}^{2m}$, we can take a neighborhood U of T in $M \times \mathbb{R}^{2m}$ in such a way that $\mathscr{H} := \tilde{\mathscr{G}}|_U \rightrightarrows U$ is Morita equivalent to the groupoid $\mathscr{N} := N \times_T N \rightrightarrows N$ associated to the submersion $N \to T$. The Lie algebroid $A\mathscr{N} = N \oplus N \cong F \oplus \mathbb{R}^{2m} \rightrightarrows N$. Let $\pi_F : F \to M$ and $p_F : N \oplus N \cong F \times \mathbb{R}^{2m} \to F$ be the obvious projections.

Definition 2.1. By the **twisted topological index of** (M, F, α) we mean the morphism

$$ind_{t,(M/F,\alpha)} : K^i\big(F^*, \alpha_0 \circ \pi_F\big) \to K^i(M/F, \alpha) := K^i(\mathcal{G}, \alpha)$$

given by the composition $K^i(F^*, \alpha_0 \circ \pi_F) \xrightarrow{Bott} K^i(F^* \times \mathbb{R}^{2m}, \alpha_0 \circ \pi_F \circ p_F) \xrightarrow{Thom^{-1}} K^i(T, \alpha_0 \circ \pi_T) \cong K^i(\tilde{\mathcal{G}}|_U, \alpha \circ p \circ j) \xrightarrow{j_!} K^i(\mathcal{G} \times \mathbb{R}^{2m}, \alpha \circ p) \xrightarrow{Bott^{-1}} K^i(\mathcal{G}, \alpha)$, where $j : \tilde{\mathcal{G}}|_U \hookrightarrow \mathcal{G} \times \mathbb{R}^{2m}$ is the inclusion as an open subgroupoid, and $p : \mathcal{G} \times \mathbb{R}^{2m} \to \mathcal{G}$ is the projection.

The first main theorem of this note is the following longitudinal twisted index theorem for $(M/F, \alpha)$ which reduces to the Connes–Skandalis index theorem for M/F in [5] when α is trivial. When the foliation is given by fibers of a fibration, our index theorem generalizes the result of Mathai–Melrose–Singer in [8] for torsion twistings.

Theorem 2.2. *For a regular foliation (M, F) with a twisting α on its holonomy groupoid M/F, we have the following equality of morphisms*:

$$ind^a_{(M/F,\alpha)} = ind^t_{(M/F,\alpha)} : K^i\big(F^*, \alpha_0 \circ \pi_F\big) \to K^i(M/F, \alpha). \tag{2}$$

Remark 1.

(i) In fact, any index morphism for foliations with twistings satisfying the three properties of Proposition 1.4 is equal to the twisted topological index.
(ii) There are two proofs of the longitudinal twisted index theorem, one adapted from Connes–Skandalis' proof in [5] from the wrong-way functoriality results in Section 3, and another one in the spirit of the index theorem proved in [3], Theorem 6.4, which is based on the fact that the Thom (inverse) isomorphism can be realized as the index of some deformation groupoid (cf. [6]).

3. Geometric cycles, push-forward maps and Baum–Connes assembly map

Let (M, F) be a foliated manifold with a twisting α on its leaf space M/F as above. For the canonical projection $p_M : M \to M/F$ which is a submersion as a Hilsum–Skandalis morphism between Lie groupoids (cf. [7]), we define the push-forward map

$$(p_M)_! : K^{n+i}(M, \alpha_0 + o_F) \to K^i(M/F, \alpha)$$

to be the composition of the Thom isomorphism $K^{n+i}(M, \alpha_0 + o_F) \cong K^i(F^*, \pi^*\alpha)$ in [1] and the analytic index morphism $ind^a_{(M/F,\alpha)} : K^i(F^*, \pi^*\alpha) \to K^i(M/F, \alpha)$ in Definition 1.3. Here o_F is the orientation twisting associated to the oriented real vector bundle F and n is the rank of F as a real vector bundle.

We now construct the push-forward map associated to any smooth map $f : W \to M/F$ and a twisting $\alpha : M/F \to PU(H)$. Let $f : W \to M/F$ be a smooth submersion which defines the pull-back foliation F_W on W, then the map f factors as $\tilde{f} \circ p_W : W \to W/F_W \to M/F$ where $\tilde{f}$ is given by the graph $\mathcal{G}_f$ seen as a $\mathcal{G}_M$-principal bundle over $\mathcal{G}_W$ with actions explicitly described by Lemma 4.2 in [5].

In the presence of a twisting $M/F \xrightarrow{\alpha} PU(H)$, denote by $\tilde{f}^*\alpha$ the pull-back twisting on W/F_W. With respect to a cover $\Omega = \{\Omega_i\}$ of M such that α is given by a strict groupoid morphism $\mathcal{G}_\Omega \to PU(H)$ as in Section 1. We have the corresponding central extensions R_M and R_W over $(M/F)_\Omega$ and $(W/F)_{f^*\Omega}$, respectively, together with generalized morphisms $\tilde{f}_\alpha : R_W \to R_M$, defined by the graph R_f. Note that R_f is an S^1-principal bundle over the graph $(\mathcal{G}_f)_\Omega$ associated to the generalized morphism between $(W/F)_{f^*\Omega}$ and $(M/F)_\Omega$.

As in [5], we can apply the actions (left and right respectively) of R_W and R_M on R_f to define a Hilbert $C^*(R_M)$-module $\mathcal{E}_{\tilde{f}_\alpha}$ (as the completion of $C_c^\infty(R_f, \Omega^{\frac{1}{2}})$) together with a representation of $C^*(R_W)$ as compact endomorphisms of $\mathcal{E}_{\tilde{f}_\alpha}$. Hence we have the Connes–Skandalis element $[\mathcal{E}_{\tilde{f}_\alpha}, 0]$ in $KK(C^*(R_W), C^*(R_M))$ and the induced morphism between the K-theory groups.

Since the map $R_W \xrightarrow{\tilde{f}_\alpha} R_M$ is S^1-equivariant, with respect to the gradation discussed in Section 1, the element $[\mathcal{E}_{\tilde{f}_\alpha}, 0]$ gives rise to a morphism, denoted by $\tilde{f}_! : K^*(W/F_W, \tilde{f}^*\alpha) \to K^*(M/F, \alpha)$. Composing with the push-forward map for the projection p_W, we define

$$f_! : K^{n_W+i}\big(W, f^*\alpha_0 + o_{F_W}\big) \to K^*(M/F, \alpha) \tag{3}$$

to be the composition $f_! = \tilde{f}_! \circ (p_W)_!$.

Definition 3.1. Let $(M/F,\alpha)$ be a twisted foliation with the normal bundle τ of F in TM, and $f: W \to M/F$ be a smooth map. Let $n_W = \dim W - rank(\tau)$. We define

$$f_!: K^{n_W+i}\big(W, f^*(\alpha_0 + o_\tau) + o_{TW}\big) \to K^i(M/F,\alpha)$$

to be the composition $g_! \circ j_!$ for any factorization $f: W \xrightarrow{j} Z \xrightarrow{g} M/F$ through a smooth submersion g, where $j_!: K^{n_W+i}(W, f^*(\alpha_0+o_\tau)+o_{TW}) \to K^{n_Z+i}(Z, g^*(\alpha_0+o_\tau)+o_{TZ}) \cong K^{n_Z+i}(Z, g^*\alpha_0 + o_{F_Z})$ is the push-forward map in twisted K-theory (cf. [1]).

Note. To justify the above definition we proceed as in Connes–Skandalis [5, Section 4] (in particular in proving the analogs of Lemma 4.7 and Proposition 4.9 in twisted cases). Our proofs are different from [5] because our index morphisms are constructed by means of deformation groupoids. The method of the proofs is to quantize the shriek maps appearing in the statements, more or less in the spirit of the proof of Theorem 6.2 in [6] where the authors deform the Thom element using a deformation groupoid (what they called the Thom groupoid). So even in the absence of twistings, what we establish is a new proof of the results of Connes–Skandalis [5, Section 4], which does not use the longitudinal pseudodifferential calculus but instead the strict deformation quantization method furnished by the Connes tangent groupoid. This was suggested by Connes in his book [4] (Section II.8.γ). The second theorem of this note is the twisted version of Theorem 4.11 in [5].

Theorem 3.2. *The push-forward map is functorial, that is, if we have a composition of smooth maps $Z \xrightarrow{g} W \xrightarrow{f} M/F$ and a twisting α on M/F, then $(f\circ g)_! = f_! \circ g_!$, where $g_!$ is the push-forward map in twisted K-theory.*

Following Connes [4], the definition of twisted geometric cycles for spaces introduced in [12], can be extended to foliations by using generalized morphisms: A geometric cycle for $(M/F;\alpha)$ is given by $(W, f, \eta, [E])$ such that W is a smooth closed manifold, $f: W \to M/F$ is a generalized morphism, η is a fix equivalence (a fixed homotopy if taking the classifying spaces) between $W \xrightarrow{o_{TW}} PU(H)$ and $W \xrightarrow{f} M/F \xrightarrow{\alpha+o_{\tau_M}} PU(H)$, where o_{TW} and o_{τ_M} are the orientation twistings associated to TW and the normal bundle τ_M of (M,F) respectively, and $[E]$ is a K-theoretical class in $K^0(W)$ represented by a $\mathbb{Z}_2$-graded vector bundle E over W.

By introducing a notion of isomorphism between cycles and imposing the Baum–Douglas equivalence relation (see [12] for more details), we get an abelian group $K_*^{geo}(M/F,\alpha+o_{\tau_M})$, the geometric twisted K-homology of $(M/F,\alpha+o_{\tau_M})$.

Theorem 3.3. *The element $\mu_\alpha(W,f,\eta,[E]) = f_!([E]) \in K^*(M/F,\alpha)$ only depends upon the equivalence class of the twisted cycle $(W,f,\eta,[E])$ in $K_*^{geo}(M/F,\alpha+o_{\tau_M})$. In particular, we have a well-defined assembly map*

$$\mu_\alpha : K_*^{geo}(M/F,\alpha+o_{\tau_M}) \to K^*(M/F,\alpha). \tag{4}$$

There is a natural geometrical index morphism $ind^g_{(M/F,\alpha)} : K^i(F^*, \alpha_0\circ\pi_F) \to K_i^{geo}(M/F,\alpha+o_{\tau_M})$ satisfying $ind^a_{(M/F,\alpha)} = \mu_\alpha \circ ind^g_{(M/F,\alpha)}$.

When α is trivial we recover the assembly map $\mu: K_*^{geo}(M/F, o_{\tau_M}) \to K^*(M/F)$ established by Connes–Skandalis in [5]. We expect that μ_α is an isomorphism whenever the untwisted assembly map is an isomorphism. Geometric cycles for $(M/F;\alpha)$ can be applied to study local index formulae in twisted cases.

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