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The div–curl lemma for sequences whose divergence and curl are compact in $W^{-1,1}$

Le lemme div–rot pour les suites dont la divergence et la boucle sont bornées dans $W^{-1,1}$

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A B S T R A C T

It is shown that $u_k \cdot v_k$ converges weakly to $u \cdot v$ if $u_k \rightharpoonup u$ weakly in L^p and $v_k \rightharpoonup v$ weakly in L^q with $p, q \in (1, \infty)$, $1/p + 1/q = 1$, under the additional assumptions that the sequences $\operatorname{div} u_k$ and $\operatorname{curl} v_k$ are compact in the dual space of $W_0^{1,\infty}$ and that $u_k \cdot v_k$ is equi-integrable. The main point is that we only require equi-integrability of the scalar product $u_k \cdot v_k$ and not of the individual sequences.

R É S U M É

On montre que $u_k \cdot v_k$ converge faiblement vers $u \cdot v$ si $u_k \rightharpoonup u$ faiblement dans L^p, $v_k \rightharpoonup v$ faiblement dans L^q, les séquences $\operatorname{div} u_k$ et $\operatorname{rot} v_k$ sont compactes dans l'espace dual de $W_0^{1,\infty}$ et $u_k \cdot v_k$ est équi-intégrable, pour $p, q \in (1, \infty)$, $1/p + 1/q = 1$. En effet, on n'utilise que l'équi-intégrabilité du produit scalaire $u_k \cdot v_k$, et non pas celle de chacune des suites.

1. Statement of the theorem

The div–curl lemma is the cornerstone of the theory of compensated compactness which was developed by Murat and Tartar in the late seventies [14,15,17–19], and is still a very active area of research [6]. In its classical form the lemma states the following: if $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ are sequences in $L^2(\Omega; \mathbb{R}^n)$ which converge weakly in $L^2(\Omega; \mathbb{R}^n)$ to u and v, respectively, and if $\operatorname{div} u_k$ is compact in $H^{-1}(\Omega)$ and $\operatorname{curl} v_k$ is compact in $H^{-1}(\Omega; \mathbb{M}^{n\times n})$, then

$$u_k \cdot v_k \rightharpoonup u \cdot v \quad \text{in } \mathcal{D}'(\Omega).$$

A natural generalization concerned sequences bounded in $L^p(\Omega; \mathbb{R}^n)$ and $L^q(\Omega; \mathbb{R}^n)$, respectively, where $p, q \in (1, \infty)$ are dual exponents, $1/p + 1/q = 1$, $\operatorname{div} u_k$ is compact in $W^{-1,p}(\Omega)$ and $\operatorname{curl} v_k$ is compact in $W^{-1,q}(\Omega; \mathbb{M}^{n\times n})$, respectively, see [15]. Important connections to Hardy spaces were established in [8], and an application to pairings between L^∞ vector fields and measures was developed in [3].

This Note is inspired by questions in nonlinear models in crystal plasticity [9] in a two-dimensional setting. The key point in this context is to prove that the determinant of the deformation gradient $\det \nabla\varphi_k$ converges to $\det \nabla\varphi$ under the

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assumption that $\nabla\varphi_k = G_k + B_k$ where $G_k \rightharpoonup \nabla\varphi$ weakly in L^2 and $B_k \to 0$ strongly in L^1. The key additional information is that $\det\nabla\varphi_k$ is compact in L^1.

Motivated by this application, we present a generalization of the div–curl lemma with very weak assumptions on $\operatorname{div} u_k$ and $\operatorname{curl} v_k$ and the additional assumption that $u_k \cdot v_k$ is equi-integrable (see the remarks after the theorem). We denote the dual of $W_0^{1,\infty}(\Omega)$ by $W^{-1,1}(\Omega)$.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with Lipschitz boundary and let $p, q \in (1,\infty)$ with $1/p + 1/q = 1$. Suppose $u_k \in L^p(\Omega;\mathbb{R}^n)$, $v_k \in L^q(\Omega;\mathbb{R}^n)$ are sequences such that*

$$u_k \rightharpoonup u \quad \text{weakly in } L^p\big(\Omega;\mathbb{R}^n\big) \quad \text{and} \quad v_k \rightharpoonup v \quad \text{weakly in } L^q\big(\Omega;\mathbb{R}^n\big), \tag{1}$$

and

$$u_k \cdot v_k \quad \text{is equi-integrable}. \tag{2}$$

Finally assume that

$$\operatorname{div} u_k \to \operatorname{div} u \quad \text{in } W^{-1,1}(\Omega) \quad \text{and} \quad \operatorname{curl} v_k \to \operatorname{curl} v \quad \text{in } W^{-1,1}\big(\Omega;\mathbb{M}^{n\times n}\big). \tag{3}$$

Then

$$u_k \cdot v_k \rightharpoonup u \cdot v \quad \text{weakly in } L^1(\Omega). \tag{4}$$

Remarks.

(i) The statement is almost classical under the stronger hypothesis that $|u_k|^p$ and $|v_k|^q$ are equi-integrable (see the lemma below). The main novelty is that here we require only that $u_k \cdot v_k$ is equi-integrable, and this is crucial for the application in [9].

(ii) The assumption that the inner product $u_k \cdot v_k$ is equi-integrable is necessary as can be seen from the one-dimensional example of a Fakir's carpet. Let $u_k = v_k$ be given on the unit interval by $u_k = \sqrt{k}\sum_{\ell=1}^{k} \chi_{[\ell/k, k^{-2}+\ell/k]}$. Then u_k converges to zero weakly in $L^2(0,1)$ and strongly in $L^1(0,1)$, but u_k^2 converges to one in the sense of distributions.

The crucial observation in the proof is the fact that given (2) we can construct modified sequences $\tilde{u}_k$ and $\tilde{v}_k$ such that $\tilde{u}_k \cdot \tilde{v}_k$ has the same weak limit as $u_k \cdot v_k$ and the sequences $|u_k|^p$ and $|v_k|^q$ are equi-integrable and therefore compact in $W^{-1,p}$ and $W^{-1,q}$, respectively. The sequences are constructed using the biting lemma [7,4] and Lipschitz truncations of Sobolev functions which originate in the work of Liu [12] and Acerbi and Fusco [1,2] and have found important applications in the vector-valued calculus of variations, see, e.g., [5,20,13].

In two dimensions, a change of variables leads to weak continuity of the determinant:

Corollary. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain with Lipschitz boundary, and let $\varphi_k \in W^{1,1}(\Omega;\mathbb{R}^2)$ be such that $\nabla\varphi_k = G^k + B^k$, with $B^k \to 0$ strongly in L^1 and $G^k \rightharpoonup G$ weakly in L^2. If the sequence $\det\nabla\varphi_k$ is equi-integrable, then $\det\nabla\varphi_k \rightharpoonup \det G$ weakly in L^1.*

2. Proofs

We begin with the proof of the lemma that shows how equi-integrability of $|u_k|^p$ leads to compactness of $\operatorname{div} u_k$. We say that a sequence $u_k \in L^p(\Omega;\mathbb{R}^n)$ is L^p-equi-integrable if there is an increasing function $\omega : [0,\infty) \to \mathbb{R}$ with $\lim_{t\to 0}\omega(t) = 0$, such that

$$\int_A |u_k|^p \,\mathrm{d}x \leqslant \omega(t) \quad \text{for all } A \subset \Omega \text{ measurable with } |A| \leqslant t. \tag{5}$$

Lemma. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz set, $1 < p < \infty$, and let $u_k \in L^p(\Omega;\mathbb{R}^n)$ be an L^p-equi-integrable sequence. If $\operatorname{div} u_k \to 0$ in $W^{-1,1}(\Omega)$, then $\operatorname{div} u_k \to 0$ in $W^{-1,p}(\Omega)$. The analogous statements hold for $\operatorname{curl} u_k$ and ∇u_k.*

Proof. Let ω be as in (5). By definition and density of $C_0^\infty(\Omega)$ in $W_0^{1,q}(\Omega)$,

$$\|\operatorname{div} u_k\|_{W^{-1,p}(\Omega)} = \sup\left\{ \int_\Omega \nabla\varphi \cdot u_k \,\mathrm{d}x\colon\ \varphi \in C_0^\infty(\Omega),\ \int_\Omega |\nabla\varphi|^q \,\mathrm{d}x \leqslant 1 \right\}, \tag{6}$$

where q is given by $1/p+1/q=1$. Fix $\varphi \in C_0^\infty(\Omega)$ with $\|\nabla\varphi\|_q \leqslant 1$ and $t>0$. By the truncation argument in [10, Lemma 4.1] or [11, Proposition A.2] there is a t-Lipschitz function $\psi \in W_0^{1,\infty}(\Omega)$ such that the measure of the set $M=\{\psi \neq \varphi$ or $\nabla\psi \neq \nabla\varphi\}$ is bounded by c_*/t^q, where c_* depends only on Ω. We decompose

$$\int_\Omega \nabla\varphi \cdot u_k \,\mathrm{d}x = \int_\Omega (\nabla\varphi - \nabla\psi)\cdot u_k \,\mathrm{d}x + \int_\Omega \nabla\psi \cdot u_k \,\mathrm{d}x. \tag{7}$$

The second term is bounded by $\|\nabla\psi\|_{L^\infty}\|\operatorname{div} u_k\|_{W^{-1,1}}$. The first term is concentrated on the set M, and by Hölder's inequality can be estimated by

$$\int_M (\nabla\varphi - \nabla\psi)\cdot u_k \,\mathrm{d}x \leqslant \left(\int_M \left(|\nabla\varphi| + t\right)^q \mathrm{d}x\right)^{1/q} \left(\int_M |u_k|^p \,\mathrm{d}x\right)^{1/p}. \tag{8}$$

The first factor is bounded by $\|\nabla\varphi\|_{L^q(M)} + |M|^{1/q} t \leqslant 1 + c_*^{1/q}$, the second by $(\omega(c_* t^{-q}))^{1/p}$ in view of the equi-integrability of the sequence $|u_k|^p$, and we conclude that

$$\|\operatorname{div} u_k\|_{W^{-1,p}(\Omega)} \leqslant \left(1 + c_*^{1/q}\right)\left(\omega\left(c_* t^{-q}\right)\right)^{1/p} + t\|\operatorname{div} u_k\|_{W^{-1,1}(\Omega)}, \tag{9}$$

with ω as in (5). The assertion follows with $t = \|\operatorname{div} u_k\|_{W^{-1,1}(\Omega)}^{-1/2}$. $\square$

Proof of the theorem. We divide the proof into four steps. The first three treat the case $u=v=0$.

Step 1. *Modification of u_k and v_k to obtain L^p- and L^q-equi-integrable sequences, respectively.* The sequence $|u_k|^p$ is bounded in L^1, and therefore the biting lemma [4,16] implies the existence of a sequence of sets $A_k \subset \Omega$ such that $|A_k| \to 0$ and, after extracting a subsequence, $|u_k|^p \chi_{\Omega\setminus A_k}$ is equi-integrable. Set $\tilde u_k = u_k \chi_{\Omega\setminus A_k}$. Since $\|\tilde u_k - u_k\|_{L^1(\Omega)} = \|u_k\|_{L^1(A_k)} \leqslant |A_k|^{1/q}\|u_k\|_{L^p(\Omega)}$ it follows that

$$\tilde u_k - u_k \to 0 \quad \text{in } L^1(\Omega). \tag{10}$$

Therefore the two sequences u_k, $\tilde u_k$ have the same weak limit (in L^p). Furthermore, $\nabla(\tilde u_k - u_k) \to 0$ in $W^{-1,1}(\Omega;\mathbb{M}^{n\times n})$, and therefore $\operatorname{div}\tilde u_k \to 0$ in $W^{-1,1}(\Omega)$. One proceeds analogously with v_k, obtains the corresponding sets B_k and a sequence $\tilde v_k = v_k \chi_{\Omega\setminus B_k}$. To conclude this step it remains to prove that $u_k \cdot v_k - \tilde u_k \cdot \tilde v_k \rightharpoonup 0$ in L^1. To see this, we observe that this expression vanishes outside of $A_k \cup B_k$, and that it equals $u_k \cdot v_k$ on this set. By equi-integrability of $u_k \cdot v_k$ and the fact that $|A_k \cup B_k| \to 0$, we conclude that $u_k \cdot v_k - \tilde u_k \cdot \tilde v_k \to 0$ in L^1.

Step 2. *Strong $W^{-1,p}$ convergence and reduction to the classical div–curl lemma.* The sequence $\tilde u_k$ is L^p-equi-integrable, and its divergence converges strongly to zero in $W^{-1,1}$. Therefore by the lemma we obtain that $\operatorname{div}\tilde u_k \to 0$ in $W^{-1,p}(\Omega)$. Analogously one shows that $\operatorname{curl}\tilde v_k \to 0$ in $W^{-1,q}(\Omega)$. By the classical div–curl lemma we then conclude that $\tilde u_k \cdot \tilde v_k \overset{*}{\rightharpoonup} 0$ in $\mathcal{D}'(\Omega)$.

Step 3. *Identification of the L^1-weak limit.* Since the sequence $u_k \cdot v_k$ is by assumption equi-integrable it has a subsequence which converges weakly in L^1. The same holds for $\tilde u_k \cdot \tilde v_k$. But the two limits are the same (Step 1) and the latter is zero (Step 2). Since the limit does not depend on the subsequence, the entire sequence converges. This concludes the proof if $u=v=0$.

Step 4. *General case.* We set $\tilde u_k = u_k - u$, $\tilde v_k = v_k - v$. Equi-integrability of the sequence $\tilde u_k \cdot \tilde v_k$ follows from $\int_A |u_k \cdot v|\,\mathrm{d}x \leqslant \|u_k\|_{L^p(\Omega)}\|v\|_{L^q(A)}$ for all $A \subset \Omega$ (and analogously for $u \cdot v_k$). By Steps 1–3, $\tilde u_k \cdot \tilde v_k \rightharpoonup 0$ weakly in $L^1(\Omega)$. The proof is concluded observing that $u_k \cdot v$ and $u \cdot v_k$ converge weakly in L^1 to $u \cdot v$. $\square$

Proof of the corollary. Let $u_k = (e_1 \cdot G^k)^\perp = (-G_{12}^k, G_{11}^k)$, $v_k = e_2 \cdot G^k = (G_{21}^k, G_{22}^k)$, so that $\det G^k = u_k \cdot v_k$. Since $G^k + B^k$ is a gradient, $\operatorname{div} u_k = \partial_1 B_{12}^k - \partial_2 B_{11}^k$, and therefore $\|\operatorname{div} u_k\|_{W^{-1,1}} \leqslant \|B_k\|_{L^1} \to 0$. The same estimate holds for $\operatorname{curl} v_k$. At this point the corollary follows from the theorem. $\square$

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