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# The shifted primes and the multidimensional Szemerédi and polynomial Van der Waerden theorems ☆

## *Translatés de l'ensemble des nombres premiers, théorème de Szemerédi multidimensionnel et théorème de Van der Waerden polynomial multidimensionnel*

Vitaly Bergelson [a], Alexander Leibman [a], Tamar Ziegler [b]

[a] *Department of Mathematics, The Ohio State University, Columbus, OH 43210, United States*
[b] *Department of Mathematics, Technion, Haifa 32000, Israel*

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ABSTRACT

In this short note we establish new refinements of multidimensional Szemerédi and polynomial Van der Waerden theorems along the shifted primes.


RÉSUMÉ

Nous présentons de nouveaux résultats du type Szemerédi multidimensionnel et Van der Waerden polynomial multidimensionnel le long des ensembles $\mathbb{P}-1$ et $\mathbb{P}+1$.


The goal of this short note is to establish new refinements of multidimensional Szemerédi and polynomial Van der Waerden theorems. Let $\mathbb{P}$ be the set of positive prime integers. We provide a short derivation of the following statements:

**Theorem 1.** *Let $\vec{m}_1, \ldots, \vec{m}_k \in \mathbb{Z}^d$ and let $E$ be of positive upper Banach density in $\mathbb{Z}^d$, namely $d^*(E) = \limsup \frac{|E\cap B|}{|B|} > 0$, where the limsup is taken over parallelepipeds $B \subset \mathbb{Z}^d$, $B = \prod_{i=1}^d [M_i, N_i]$ with $\min_i |N_i - M_i| \to \infty$. Then the set*

$$R(E) = \{n \in \mathbb{N}\colon d^*(E \cap (E - n\vec{m}_1) \cap \cdots \cap (E - n\vec{m}_k)) > 0\}$$

*has a nonempty intersection with $\mathbb{P}-1$ and with $\mathbb{P}+1$.*

**Theorem 2.** *Let $(X, \mathcal{B}, \mu)$ be a finite measure space and let $T_1, \ldots, T_k$ be pairwise commuting measure preserving transformations of $X$. Let $A \in \mathcal{B}$, $\mu(A) > 0$; then the set*

$$R(A) = \{n \in \mathbb{N}\colon \mu(A \cap T_1^{-n} A \cap \cdots \cap T_k^{-n} A) > 0\}$$

*has a nonempty intersection with $\mathbb{P}-1$ and with $\mathbb{P}+1$.*

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*E-mail addresses:* vitaly@math.ohio-state.edu (V. Bergelson), leibman@math.ohio-state.edu (A. Leibman), tamarzr@tx.technion.ac.il (T. Ziegler).

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Theorems 1 and 2 are equivalent via the Furstenberg correspondence principle. (See, for example, Theorem 6.4.17 in [1] or Theorem 2.1 in [4].) We also remark that for any integer $a \neq \pm 1$ one can easily construct counter examples via periodic sets/systems, so that Theorems 1 and 2 do not hold true for $\mathbb{P}+a$.

$\mathrm{IP}_r$ and $\mathrm{IP}_r^*$ sets (in $\mathbb{N}$) are defined as follows:

**Definition.** For $r \in \mathbb{N}$, *an $IP_r$ set* in $\mathbb{N}$ is a set of the form $\{\vec{n} \cdot \vec{w}\}_{\vec{w} \in \{0,1\}^r \setminus \{\vec{0}\}}$, where $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. A subset of $\mathbb{N}$ is *an $IP_r^*$ set* if it has a nonempty intersection with every $\mathrm{IP}_r$ set in $\mathbb{N}$.

For example, an $\mathrm{IP}_3$ set is a 7-element set of the form $\{n, m, k, n+m, n+k, m+k, n+m+k\} \subset \mathbb{N}$.

Our proof of Theorem 1 is based on the following two very deep theorems. The first was obtained by Furstenberg and Katznelson in [6] (see Theorem 10.1 and the remark on page 168):

**Theorem 3.** *For any probability measure space $(X, \mathcal{B}, \mu)$, any commuting measure preserving transformations $T_1, \ldots, T_k$, and any set $A \in \mathcal{B}$ of positive measure, there exists an integer r such that the set $R(A)$ is an $IP_r^*$ set.*

The second was obtained in a series of papers by Green, Tao and Ziegler in [8–10].

**Theorem 4.** *Let $\psi_1, \ldots, \psi_l$ be affine linear forms in r variables with integer coefficients, $\psi_i(\vec{x}) = \sum_{j=1}^{r} m_{i,j} x_j + c_i$, no two of which are affinely dependent. Then there exists an $\vec{n} \in \mathbb{Z}^r$ such that $\psi_1(\vec{n}), \ldots, \psi_l(\vec{n}) \in \mathbb{P}$ iff for any $k \in \mathbb{N}$, $k \geqslant 2$, there exists an $\vec{x} \in \mathbb{Z}^r$ such that $\psi_1(\vec{x}), \ldots, \psi_l(\vec{x})$ are all nondivisible by k.*

As a corollary, we get that the set $\mathbb{P}-1$ (as well as the set $\mathbb{P}+1$) contains an $\mathrm{IP}_r$ set in $\mathbb{N}$ for every $r \in \mathbb{N}$. Indeed, for any $r$, since 1 is not divisible by any $k \geqslant 2$, by Theorem 4 there exists $\vec{n} \in \mathbb{Z}^r$ such that the integers $\vec{w} \cdot \vec{n} + 1$, $\vec{w} \in \{0,1\}^r \setminus \{\vec{0}\}$, are all prime.

**Proof of Theorem 2.** By Theorem 3, $R(A)$ nontrivially intersects any $\mathrm{IP}_r$ set in $\mathbb{N}$ for $r$ large enough, and by Theorem 4, the sets $\mathbb{P}-1$ and $\mathbb{P}+1$ contain $\mathrm{IP}_r$ sets for all $r$. □

We remark that in the case $d=1$ and $T_i = T^i$, $i = 1, \ldots, k$, Theorems 1, 2 were proved in [5] conditional on the *inverse conjecture for the Gowers norms* which was recently obtained in [10]. However, in their full generality, Theorems 1, 2 cannot be obtained by the methods in that paper.

We also remark that one cannot obtain polynomial extensions of Theorems 1 and 2 by the methods of the present short note, since there is so far no polynomial version of Theorem 3. (See however, [12], where such an extension has been obtained for the case $T_i$ are all equal to the same transformation.) On the other hand, a "partition" version of Theorem 1 (and of the topological version of Theorem 2) can be extended to polynomials:

**Theorem 5.** *For any $d \in \mathbb{N}$ and any finite partition $\mathbb{Z}^d = \bigcup_{s=1}^{c} C_s$, at least one of the sets $C_s$ has the property that for any finite set of polynomials $\vec{f}_i : \mathbb{Z} \to \mathbb{Z}^d$, $i = 1, \ldots, k$, satisfying $\vec{f}_i(0) = 0$ for all i, there exist $p \in \mathbb{P}$ and $\vec{a} \in \mathbb{Z}^d$ such that*

$$\vec{a}, \vec{a} + \vec{f}_1(p-1), \ldots, \vec{a} + \vec{f}_k(p-1) \in C_s,$$

*and there exist $q \in \mathbb{P}$ and $\vec{b} \in \mathbb{Z}^d$ such that*

$$\vec{b}, \vec{b} + \vec{f}_1(q+1), \ldots, \vec{b} + \vec{f}_k(q+1) \in C_s.$$

A parallel topological dynamical result is the following refinement of Theorem $C$ in [2]:

**Theorem 6.** *Let $(X, \rho)$ be a compact metric space and let $T(\vec{m})$, $\vec{m} \in \mathbb{Z}^d$, be an action of $\mathbb{Z}^d$ on X by continuous transformations. Then for any finite set of polynomials $\vec{f}_i : \mathbb{Z} \to \mathbb{Z}^d$, $i = 1, \ldots, k$, with $\vec{f}_i(0) = 0$ for all i, and any $\varepsilon > 0$ there exist a point $x \in X$ and a prime integer $p \in \mathbb{P}$ such that $\rho(x, T(\vec{f}_i(p-1))x) < \varepsilon$ for all $i = 1, \ldots, k$, and there exist a point $y \in X$ and a prime integer $q \in \mathbb{P}$ such that $\rho(y, T(\vec{f}_i(q+1))y) < \varepsilon$ for all $i = 1, \ldots, k$.*

(See [7] and [11] for a discussion of equivalence of Ramsey-theoretical and topological-dynamical results.)

The proof of Theorem 5 is the same as of Theorem 1, based on the following version of the polynomial Van der Waerden theorem:

**Theorem 7.** *For any partition $\mathbb{Z}^d = \bigcup_{s=1}^{c} C_s$ at least one of the sets $C_s$ has the property that for any finite set of polynomials $\vec{f}_i : \mathbb{Z} \to \mathbb{Z}^d$, $i = 1, \ldots, k$, with $\vec{f}_i(0) = 0$ for all i,*

$$\left\{n \in \mathbb{N}\colon \vec{a}, \vec{a} + \vec{f}_1(n), \ldots, \vec{a} + \vec{f}_k(n) \in C_s \text{ for some } \vec{a} \in \mathbb{Z}^d\right\}$$

*is an $IP_r^*$ set for r large enough.*

**Remark.** The polynomial Van der Waerden theorem, proved in [2], was formulated in a slightly weaker form: it was only claimed there (see [2], Corollary 1.12) that the set

$$\{n \in \mathbb{N}\colon\ \vec{a}, \vec{a} + \vec{f}_1(n), \ldots, \vec{a} + \vec{f}_k(n) \in C_s \text{ for some } \vec{a} \in \mathbb{Z}^d \text{ and some } s\}$$

is an IP* set. (*An IP* set* in $\mathbb{N}$ is a set that has a nonempty intersection with every IP set, where *an IP set* is an $\mathrm{IP}_\infty$ set, that is, a set of the form $\bigcup_{k=1}^{\infty}\{\vec{n} \cdot \vec{w}\}_{\vec{w} \in \{0,1\}^k \setminus \{\vec{0}\}}$ for some $\vec{n} = (n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$.) However, Theorem 7 can be easily derived from the results of [2]. Namely, the proof of Corollary 1.9 in [2] actually shows that the set $P$ in the formulation of this corollary is an $\mathrm{IP}_r^*$ set for $r$ large enough, and a standard application of this corollary to a minimal closed shift-invariant subset of the space of $c$-partitions of $\mathbb{Z}^d$ gives the desired result. Another way to get it is by utilizing the polynomial Hales–Jewett theorem [3].

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