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Number Theory

Bounds on oscillatory integral operators

Estimées sur les integrales oscillatoires

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ABSTRACT

We present new estimates in E. Stein's Fourier restriction problem for curved hyper-surfaces in $\mathbb{R}^n$ and also on the mapping properties of the more general class of oscillatory integral operators introduced by L. Hörmander.

RÉSUMÉ

Nous présentons de nouvelles estimations dans le problème de E. Stein sur la restriction de Fourier à des hyper-surfaces à courbure dans $\mathbb{R}^n$ ainsi que sur les intégrales oscillatoires introduites par L. Hörmander.

Version française abrégée

Soit $S \subset \mathbb{R}^n$ $(n \geqslant 3)$ une hyper-surface compacte et lisse et dont la seconde forme fondamentale est positivement définie. Soit σ sa mesure de surface. Pour $p > 2$ fixé et $R \to \infty$, dénotons

$$Q_R^{(p)} = \max \|\hat{\mu}\|_{L^p(B_R)} \tag{1}$$

où $B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$,

$$\hat{\mu}(\xi) = \int e^{2\pi i x.\xi} \mu(\mathrm{d}x) \tag{2}$$

et le maximum est pris sur toutes les measures μ sur S, telles que $\mu \ll \sigma$ et $\|\frac{\mathrm{d}\mu}{\mathrm{d}\sigma}\|_\infty \leqslant 1$. On a l'estimée

$$Q_R^{(p)} \ll R^\varepsilon \quad \text{pour toute } \varepsilon > 0 \tag{3}$$

si p satisfait la condition

$$\begin{cases} p \geqslant 2\dfrac{4n+3}{4n-3} & \text{si } n \equiv 0 \ (\text{mod } 3), \\ p \geqslant \dfrac{2n+1}{n-1} & \text{si } n \equiv 1 \ (\text{mod } 3), \\ p \geqslant \dfrac{4(n+1)}{2n-1} & \text{si } n \equiv 2 \ (\text{mod } 3). \end{cases} \tag{4}$$

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Pour $n = 3$, on a (3) pour $p \geqslant 3,3$.

Considérons ensuite des intégrales oscillatoires de la forme

$$(T_\lambda f)(x) = \int_{\text{loc}} e^{i\lambda\psi(x,y)} f(y)\,\mathrm{d}y \tag{5}$$

où x (resp. y) sont dans un voisinage de $0 \in \mathbb{R}^n$ (resp. $0 \in \mathbb{R}^{n-1}$).

La fonction de phase $\psi(x, y)$ a la forme

$$\psi(x,y) = x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n \langle Ay, y\rangle + 0\big(|x||y|^3\big) + 0\big(|x|^2|y|^2\big) \tag{6}$$

où A est non-dégénéré.

Si n est pair et $\lambda \to \infty$, on a l'estimation

$$\|T_\lambda f\|_p \ll \lambda^{-\frac{n}{p}+\varepsilon} \|f\|_\infty \quad \text{pour tout } \varepsilon > 0 \text{ et } p \geqslant \frac{2(n+2)}{n}. \tag{7}$$

En supposant A positivement (ou négativement) défini, l'inégalité (7) est vrai si p satisfait les conditions (4) ($n \geqslant 3$ arbitraire).

1. Fourier transform of measures carried by curved hyper-surfaces

Let $S \subset \mathbb{R}^n$ be a smooth, compact hyper-surface with positive definite second fundamental form and let σ be its surface measure (the sphere $S = S^{(n-1)}$ and the paraboloid $(y, |y|^2) \subset \mathbb{R}^n$ are the most important model cases). Denote

$$Q_R^{(p)} = \max \|\hat{\mu}\|_{L^p(B_R)}$$

where $B_R = \{x \in \mathbb{R}^n;\ |x| \leqslant R\}$ and the maximum is taken over all measures $\mu \ll \sigma$ on S such that $\|\frac{\mathrm{d}\mu}{\mathrm{d}\sigma}\|_\infty \leqslant 1$. We have

Theorem 1.

$$Q_R^{(p)} \ll R^\varepsilon \quad \textit{for all } \varepsilon > 0 \tag{8}$$

provided

$$\begin{cases} p \geqslant 2\dfrac{4n+3}{4n-3} & \textit{if } n \equiv 0 \ (\mathrm{mod}\ 3), \\ p \geqslant \dfrac{2n+1}{n-1} & \textit{if } n \equiv 1 \ (\mathrm{mod}\ 3), \\ p \geqslant \dfrac{4(n+1)}{2n-1} & \textit{if } n \equiv 2 \ (\mathrm{mod}\ 3) \end{cases} \tag{9}$$

and

Theorem 2. *For $n = 3$, (8) holds for $p \geqslant 3,3$.*

Previous best results were due to T. Tao [4], based on an L^2-bilinear estimate (going back to the work of T. Wolff), providing the bound $Q_R^{(p)} < C_p$ for $p > \frac{2(n+2)}{n}$. Thus, apart from the R^ε-factor, we improve the exponent in all dimensions, except $n = 4$.

Let us recall E. Stein's conjecture, stating that $Q_R^{(p)} < C_p$ for $p > \frac{2n}{n-1}$ and which presently is only known to hold for $n = 2$.

2. Oscillatory integrals of Hörmander type

We consider oscillatory integral operators of the form

$$(T_\lambda f)(x) = \int_{\text{loc}} e^{i\lambda\psi(x,y)} f(y)\,\mathrm{d}y$$

with real analytic phase function

$$\psi(x,y) = x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n \langle Ay, y\rangle + 0\big(|x||y|^3\big) + 0\big(|x|^2|y|^2\big) \tag{10}$$

and $A \in \mathrm{Mat}_{n-1}(\mathbb{R})$ non-degenerate.

Here $x \in \mathbb{R}^n$ (resp. $y \in \mathbb{R}^{n-1}$) are restricted to sufficiently small neighborhoods of 0 and $\lambda \to \infty$ is a parameter. Note that if ψ is linear in x, we recover the Fourier transform of a hyper-surface carried measure as considered in Section 1.

We are interested in the mapping properties of T_λ. Recall the important L^2-inequality (cf. [3])

$$\|T_\lambda f\|_p \leqslant c\lambda^{-\frac{n}{p}} \|f\|_2 \quad \text{for } p \geqslant \frac{2(n+1)}{n-1}. \tag{11}$$

For $n = 2$, Hörmander (providing an alternative proof to a result on Bochner–Riesz multipliers, due to Carleson and Sjolin) showed in particular that

$$\|T_\lambda f\|_p \leqslant C\lambda^{-\frac{2}{p}} \|f\|_\infty \quad \text{for } p > 4 \tag{12}$$

and raised the question of its higher dimensional generalization for $p > \frac{2n}{n-1}$. Surprisingly (cf. [2]), the answer turned out to be negative. For n odd, there are examples of phase functions ψ such that an inequality of the form

$$\|T_\lambda f\|_p \leqslant C\lambda^{-\frac{n}{p}} \|f\|_\infty \tag{13}$$

only holds for $p \geqslant \frac{2(n+1)}{n-1}$. It was also observed in [2] that this extreme situation cannot occur for n even.

Recently, we proved the following:

Theorem 3. *For n even, $\lambda \to \infty$*

$$\|T_\lambda f\|_p \ll \lambda^{-\frac{n}{p}+\varepsilon} \|f\|_\infty \quad \textit{for } p \geqslant \frac{2(n+2)}{n}$$

and apart from the λ^ε-factor, Theorem 3 *as a general statement is best possible.*

Next, let us specify (10) further by requiring A to be positive (or negative) definite.

Theorem 4. ($n = 3$)*. For $p > \frac{10}{3}$, assuming A positive definite and ψ a polynomial, one has the inequality*

$$\|T_\lambda f\|_p \leqslant C\lambda^{-\frac{3}{p}} \|f\|_\infty$$

and there are such examples where the result is best possible (*apart from the endpoint*)*.*

Theorem 5. (*n arbitrary*)*. Assuming A positive definite, the inequality*

$$\|T_\lambda f\|_p \ll \lambda^{-\frac{n}{p}+\varepsilon} \|f\|_\infty \quad \textit{for all } \varepsilon > 0$$

holds, for p satisfying (9)*.*

3. Comments on the method

The main ingredient in our analysis is the multilinear inequality from [1]. We briefly recall the result. Given ψ as in (10), consider the vectors

$$Z = Z(x, y) = \partial_{y_1}(\nabla_x \psi) \wedge \cdots \wedge \partial_{y_{n-1}}(\nabla_x \psi). \tag{14}$$

Fix $2 \leqslant k \leqslant n$ and open sets $U_1, \ldots, U_k$ in the y-domain, such that

$$\left|Z\big(x, y^{(1)}\big) \wedge \cdots \wedge Z\big(x, y^{(k)}\big)\right| > c \tag{15}$$

for some $c > 0$, for all x in the specified neighborhood V of $0 \in \mathbb{R}^n$ and $y^{(1)} \in U_1, \ldots, y^{(k)} \in U_k$. Then there is the inequality for $p = \frac{2k}{k-1}$

$$\left\| \left(\prod_{i=1}^{k} |T_\lambda f_i| \right)^{\frac{1}{k}} \right\|_{L^p(V)} \ll \lambda^{-\frac{n}{p}+\varepsilon} \left(\prod_{i=1}^{k} \|f_i\|_2 \right)^{\frac{1}{k}} \tag{16}$$

assuming $\operatorname{supp} f_i \subset U_i$ $(1 \leqslant i \leqslant k)$.

Note that if ψ is linear in x, $Z = Z(y)$ and assumption (15) is a transversality condition for the normal vectors of the corresponding hyper-surface.

Next, we give a sketch of the proof of Theorem 1 for $n = 3$ and taking for S the paraboloid $(y_1, y_2, \frac{1}{2}(y_1^2 + y_2^2))$. The argument contains the essence of our method. Thus

$$\psi(x, y) = x_1 y_1 + x_2 y_2 + \frac{1}{2} x_3 \big(y_1^2 + y_2^2\big) \quad \text{and} \quad Z(y) = (-y_1, -y_2, 1).$$

Condition (15) for $k=3$ amounts thus to the non-collinearity of $y^{(1)}, y^{(2)}, y^{(3)} \in \mathbb{R}^2$. Let y range in $\Omega = [|y_1|, |y_2| < 0(1)] \subset \mathbb{R}^2$. Fix large parameters $1 \ll K_1 \ll K$ and let $\{Q'_\alpha\}$ (resp. $\{Q_\beta\}$) be partitions of Ω in $\frac{1}{K_1}$ (resp. $\frac{1}{K}$) boxes. Denoting ξ_β the center of Q_β, write

$$(Tf)(x) = \int_\Omega e^{i\psi(x,y)} f(y)\,\mathrm{d}y = \sum_\beta \Big[\int_{Q_\beta} e^{i[\psi(x,y)-\psi(x,\xi_\beta)]} f(y)\,\mathrm{d}y\Big] e^{i\psi(x,\xi_\beta)} = \sum_\beta (T_\beta f)(x) e^{i\psi(x,\xi_\beta)}. \tag{17}$$

Fix a ball $B(a,K) \subset B_R \subset \mathbb{R}^3$. Roughly speaking, we may view $T_\beta(x)$ as essentially a constant c_β on $B(a,K)$ and denote $c_* = \max |c_\beta|$. We distinguish two alternatives.

(i) There are (non-collinear) boxes $Q_{\beta_1}, Q_{\beta_2}, Q_{\beta_3}$ such that

$$\big|\big(y^{(1)} - y^{(2)}\big) \wedge \big(y^{(1)} - y^{(3)}\big)\big| > \frac{1}{K^2} \quad \text{for } y^{(i)} \in Q_{\beta_i}\ (1 \leqslant i \leqslant 3) \tag{18}$$

and

$$|c_{\beta_i}| > K^{-2} c_* \quad \text{for } i = 1,2,3. \tag{19}$$

(ii) The negation of (i), implying that there is a line segment $\ell \subset \Omega$ such that

$$|c_\beta| < K^{-2} c_* \quad \text{if } \operatorname{dist}(Q_\beta, \ell) \geqslant 10K^{-1}.$$

Assume (i). From (17), (19), $|Tf(x)| \lesssim K^2 c_* \lesssim K^4 (|T_{\beta_1} f|.|T_{\beta_2} f|.|T_{\beta_3} f|)^{1/3}(x)$. The contribution to $L^p(B_R)$ is bounded by

$$K^4 \Big\{ \sum_{\beta_1,\beta_2,\beta_3 (18)} \big\| (T_{\beta_1} f . T_{\beta_2} f . T_{\beta_3} f)^{1/3} \big\|^p_{L^p(B_R)} \Big\}^{\frac{1}{p}} \ll C(K) R^\varepsilon \tag{20}$$

for $p \geqslant 3$, by the [1] 3-linear L^3-bound, cf. (16).

If (ii), proceed as follows. Considering the partition $\{Q'_\alpha\}$ of Ω, write similarly to (17), $Tf(x) = \sum_\alpha (T'_\alpha f)(x) e^{i\psi(x,\xi'_\alpha)}$. Fix $x \in B(a,K)$. Either

$$\big|Tf(x)\big| \leqslant 10^3 \max_\alpha \big|T'_\alpha f(x)\big| \tag{21}$$

or there are α_1, α_2 satisfying

$$\operatorname{dist}\big(Q'_{\alpha_1}, Q'_{\alpha_2}\big) > \frac{10}{K_1} \tag{22}$$

and

$$\big|\big(T'_{\alpha_1} f\big)(x)\big|, \big|\big(T'_{\alpha_2} f\big)(x)\big| \gtrsim \frac{1}{K_1^2} \big|Tf(x)\big|. \tag{23}$$

Using parabolic rescaling, the contribution of (21) is estimated by

$$\Big(\sum_\alpha \|T'_\alpha f\|^p_{L^p(B_R)} \Big)^{1/p} \lesssim K_1^{2/p} K_1^{-2+4/p} Q^{(p)}_{R/K_1} \leqslant K_1^{-2+6/p} Q^{(p)}_R. \tag{24}$$

Next, assume (22), (23). Note that on $B(a,K)$ by (ii)

$$\big|\big(T'_\alpha f\big)(x)\big| \leqslant \Big| \sum_{\substack{Q_\beta \subset Q'_\alpha \\ \operatorname{dist}(Q_\beta,\ell) < 10K^{-1}}} c_\beta e^{i\psi(x,\xi_\beta)} \Big| + \max_\beta \big|T_\beta f(x)\big|.$$

Hence, by (23), either

$$\big|Tf(x)\big| \lesssim K_1^2 \max_\beta \big|T_\beta f(x)\big| \tag{25}$$

or

$$\big|Tf(x)\big| \lesssim K_1^2 \max_{\alpha_1,\alpha_2 (22)} \Big| \sum_{\substack{Q_\beta \subset Q'_{\alpha_1} \\ \operatorname{dist}(Q_\beta,\ell) < 10K^{-1}}} c_\beta e^{i\psi(x,\xi_\beta)} \Big|^{\frac{1}{2}} \Big| \sum_{\substack{Q_\beta \subset Q'_{\alpha_2} \\ \operatorname{dist}(Q_\beta,\ell) < 10K^{-1}}} \cdots \Big|^{\frac{1}{2}}. \tag{26}$$

The $L^p(B_R)$ contribution of (25) is bounded by, cf. (24)

$$K_1^2 K^{\frac{6}{p}-2} Q_R^{(p)}. \tag{27}$$

Using a bilinear L^4-bound, estimate

$$\begin{aligned}\|(26)\|_{L^p(B(a,K))} &\leqslant K^{3(\frac{1}{p}-\frac{1}{4})}\|(26)\|_{L^4(B(a,K))} \leqslant C(K_1)K^{\frac{3}{p}}\Bigg(\sum_{\mathrm{dist}(Q_\beta,\ell)<10K^{-1}} |c_\beta|^2\Bigg)^{\frac{1}{2}} \\ &< c(K_1)K^{\frac{1}{2}+\frac{2}{p}}\Bigg(\sum_\beta \big|T_\beta f(x)\big|^p\Bigg)^{\frac{1}{p}}\end{aligned}$$

and integrating on B_R, we obtain

$$C(K_1)K^{\frac{1}{2}-\frac{1}{p}}\Bigg(\sum_\beta \|T_\beta f\|^p_{L^p(B_R)}\Bigg)^{\frac{1}{p}} < C(K_1)K^{\frac{5}{p}-\frac{3}{2}} Q_R^{(p)}. \tag{28}$$

In view of (24), (27), (28) and (20), a suitable choice of $K_1 \ll K$ shows that indeed $Q_R^{(p)} \ll R^\varepsilon$ for $p > \frac{10}{3}$.

Remark. Theorems 1, 3, 4 and 5 are obtained by generalizing and refining the above technique. The proof of Theorem 2 relies moreover on T. Wolff's estimate for the Kakeya maximal function [5].

4. Curved Kakeya sets

A Kakeya set in $\mathbb{R}^n$ is a compact set $A \subset \mathbb{R}^n$ containing a unit line segment in every direction. Recall that E. Stein's conjecture in Section 1 implies that such sets have maximal Minkowski dimension (i.e. $= n$). Presently, the 'Kakeya conjecture' remains open for $n \geqslant 3$.

Similarly, the mapping properties of T_λ in Section 2 are closely related to the structure of the associated 'curved Kakeya sets', which are compact sets $A \subset \mathbb{R}^n$ containing a curve $\Gamma_y = [\nabla_y \psi = b(y)]$ for each $y \in \Omega$. Here $b : \Omega \to B_1 \subset \mathbb{R}^n$ is arbitrary. For n odd, the dimension of such sets may be as small as $\frac{n+1}{2}$. Taking $n = 3$, the 2D-compression phenomenon may occur in either the hyperbolic or elliptic setting, as illustrated for instance by the following phase functions

$$\psi_1(x, y) = -x_1 y_1 - x_2 y_2 + 2x_3 y_1 y_2 + x_3^2 y_2^2, \tag{29}$$

$$\psi_2(x, y) = -x_1 y_1 - x_2 y_2 + x_3\left(\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2\right) + x_3^2 y_1 y_2 + \frac{1}{2}x_3^3 y_2^2. \tag{30}$$

Let us point out that despite this similarity, ψ_1 violates inequality (13) whenever $p < 4$ (see [2]), while for ψ_2, by Theorem 4, (13) holds for $p > \frac{10}{3}$ (and fails for $p < \frac{10}{3}$).

In even dimension, there is the following result, providing a sharp version of a phenomenon first observed in [2], and which in some sense is the companion to Theorem 3.

Theorem 6. *For n even and ψ as in* (10)*, any curved Kakeya set has Minkowski dimension at least* $\frac{n}{2} + 1$.

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