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Number Theory

Reciprocity formulae for multiple Dedekind–Rademacher sums

Lois de réciprocité pour les sommes multiples de Dedekind–Rademacher

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ABSTRACT

We introduce multiple Dedekind–Rademacher sums, in terms of values of Bernoulli functions, that generalize the classical Dedekind–Rademacher sums. The aim of this paper is to give and prove a reciprocity law for these sums. The main theorem presented in this paper contains all previous results in the literature about Dedekind–Rademacher sums.

RÉSUMÉ

Nous introduisons les sommes multiples de Dedekind–Rademacher, écrites en termes de valeurs des fonctions de Bernoulli. Ses sommes généralisent les sommes classiques de Dedekind–Rademacher. Dans ce travail, nous avons établi une loi de réciprocité pour ces sommes. Notre résultat unifie et généralise tous les résultats connus sur les sommes de Dedekind–Rademacher.

Version française abrégée

Les sommes de Dedekind classiques interviennent dans divers domaines des mathématiques. En particulier, en théorie des nombres [3,6,8,10,13,15], géométries algébrique, différentielle et combinatoire [1,3,5], topologie [2,7,11]. L'objectif de ce travail est d'établir les lois de réciprocité satisfaites par les sommes multiples de Dedekind–Rademacher. Celles-ci généralisent la fameuse loi de réciprocité de Dedekind (3). Soient $n \in \mathbb{N}$, $n \geqslant 2$, $a_1, \dots, a_n, r_1, \dots, r_k, \dots, r_n \in \mathbb{N}$. Pour $1 \leqslant k \leqslant n$ on note $\overrightarrow{A_k} = (a_1, \dots, \check{a}_k, \dots, a_n)$, $\overrightarrow{R_k} = (r_1, \dots, \check{r}_k, \dots, r_n) \in \mathbb{N}^{n-1}$, où $\check{x}_k$ signifie que l'on omet le terme x_k. On définit les sommes multiples de Dedekind–Rademacher par

$$S_n(\overrightarrow{A_k}, \overrightarrow{R_k}) = \sum_{t=0}^{a_k-1} \prod_{1 \leqslant j \neq k \leqslant n} \bar{B}_{r_j}\left(\frac{a_j t}{a_k}\right).$$

Lorsque $n = 3$ on retrouve les sommes classiques de Dedekind et Rademacher. On introduit la série génératrice suivante :

$$\mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) := \sum_{\overrightarrow{R_k} \in \mathbb{N}^{n-1}} \frac{S_n(\overrightarrow{A_k}, \overrightarrow{R_k})}{r_1! \dots \check{r}_k! \dots r_n!} \prod_{1 \leqslant j \neq k \leqslant n} \left(\frac{2\pi i \varphi_j}{a_j}\right)^{r_j - 1}$$

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où $\overrightarrow{\Phi_k} = (\phi_1, \ldots, \check{\phi}_k, \ldots, \phi_n)$ est un $(n-1)$-uplet de réels et les $\overline{B}_n(z)$ désignent habituellement les fonctions de Bernoulli (voir paragraphe 1). Le résultat principal de cette Note est le théorème suivant :

Théorème 1. *Soient n un entier naturel $\geqslant 2$, $a_1, \ldots, a_n$ entiers naturels non nuls deux à deux premiers entre eux et $\varphi_1, \ldots, \varphi_n$ n variables réelles dans $]-1, 1[$, dont la somme vaut zéro. Alors*

$$\sum_{k=1}^{n} \mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) = -\frac{1}{(2i)^{n-1}} \frac{\sin(\pi \sum_{j=1}^{n} \frac{\varphi_j}{a_j})}{\prod_{j=1}^{n} \sin(\pi \frac{\varphi_j}{a_j})} + \frac{1}{(2i)^{n-1}} \sum_{k=1}^{n} \prod_{1 \leqslant j \neq k \leqslant n} \cot\left(\pi \frac{\varphi_j}{a_j}\right).$$

La preuve de ce résultat est détaillée dans le paragraphe 2. Lorsque $n = 3$ on retrouve les théorèmes de Dedekind (3), Rademacher (5) et Hall–Wilson–Zagier [6].

1. Introduction and statement of main results

Dedekind sums appear in many different areas of mathematics such as number theory [4,6,8,10,13,15], cohomology [9, 14], algebraic, differential and combinatorial geometries [1,3,5], topology [2,7,11]. We fix the notations: $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^* = \{1, 2, \ldots\}$. Let us recall the Bernoulli polynomials and the classical Dedekind sum. The Bernoulli polynomials are defined by the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}, \quad |z| < 2\pi; \tag{1}$$

the Bernoulli function $\overline{B}_n(z)$ is defined by the unique function which is periodic of period 1 and coincides with $B_n(z)$ on $[0, 1[$ (except that we set $\overline{B}_1(z) = 0$ for $z \in \mathbb{Z}$).

If a, b are coprime integers, the classical Dedekind sum $S(a, b)$ is defined by

$$S(a, b) = \sum_{t=0}^{a-1} \overline{B}_1\left(\frac{bt}{a}\right) \overline{B}_1\left(\frac{t}{a}\right). \tag{2}$$

Dedekind [4] introduced this sum in connection with the transformation formula for the Dedekind η-function and deduced from this his reciprocity formula

$$S(a, b) + S(b, a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right). \tag{3}$$

The Dedekind sum has been generalized, notably by Rademacher [13], who considered a new integer $c > 0$ and introduced the homogeneous sums

$$S(a, b, c) := \sum_{t=0}^{a-1} \overline{B}_1\left(\frac{bt}{a}\right) \overline{B}_1\left(\frac{ct}{a}\right). \tag{4}$$

Rademacher's reciprocity formula (which is not implied by Dedekind's) is given by:

$$S(a, b, c) + S(b, c, a) + S(c, a, b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{c}{ab} + \frac{b}{ac}\right) \quad (a, b, c \text{ pairwise coprime}). \tag{5}$$

For fixed integers $n \geqslant 2$, we consider $(a_1, \ldots, a_n) \in (\mathbb{N}^*)^n$; $(r_1, \ldots, r_k, \ldots, r_n) \in \mathbb{N}^n$ and we set for any $1 \leqslant k \leqslant n$

$$\overrightarrow{A_k} = (a_1, \ldots, \check{a}_k, \ldots, a_n), \qquad \overrightarrow{R_k} = (r_1, \ldots, \check{r}_k, \ldots, r_n) \in \mathbb{N}^{n-1}$$

where $\check{x}_k$ means we omit the term x_k. We define multiple Dedekind–Rademacher sums by the formula:

$$S_n(\overrightarrow{A_k}, \overrightarrow{R_k}) := \sum_{t=0}^{a_k-1} \prod_{1 \leqslant j \neq k \leqslant n} \overline{B}_{r_j}\left(\frac{a_j t}{a_k}\right). \tag{6}$$

Our aim here is to prove that these sums satisfy certain reciprocity relations under cyclic permutations of $(a_1, \ldots, a_k, \ldots, a_n)$. However, these reciprocity relations mix various multi-indices $\overrightarrow{R_k} = (r_1, \ldots, \check{r}_k, \ldots, r_n)$, so they are most conveniently stated in terms of the generating function formally defined by

$$\mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) := \sum_{\overrightarrow{R_k} \in \mathbb{N}^{n-1}} \frac{S_n(\overrightarrow{A_k}, \overrightarrow{R_k})}{r_1! \cdots \check{r}_k! \cdots r_n!} \prod_{1 \leqslant j \neq k \leqslant n} \left(\frac{2\pi i \varphi_j}{a_j}\right)^{r_j - 1} \tag{7}$$

where $\overrightarrow{\Phi_k} = (\phi_1, \ldots, \phi_n)$ is a $(n-1)$-uples of reals. In this Note we prove the following result

Theorem 1. *Let n be fixed integer* $\geqslant 2$, *let* $a_1, \ldots, a_n$ *be positive integers and pairwise coprime and let* $\varphi_1, \ldots, \varphi_n$ *n-real variables in the interval* $]-1, 1[$, *with sum zero. Then*

$$\sum_{k=1}^{n} \mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) = -\frac{1}{(2i)^{n-1}} \frac{\sin(\pi \sum_{j=1}^{n} \frac{\varphi_j}{a_j})}{\prod_{j=1}^{n} \sin(\pi \frac{\varphi_j}{a_j})} + \frac{1}{(2i)^{n-1}} \sum_{k=1}^{n} \prod_{1 \leqslant j \neq k \leqslant n} \cot\left(\pi \frac{\varphi_j}{a_j}\right). \tag{8}$$

Our Theorem 1 recovers Dedekind's and Rademacher's formulas (3), (5) and the theorem of Hall, Wilson and Zagier in [6] in the interesting case when the additive terms are zero. Precisely, we have the following relations deduced directly from the last theorem using the expansion of $\sin(\sum_{j=1}^{n} U_j)$ for $n = 2, 3, 4, 5,$ we obtain:

$$\sum_{k=1}^{2} \mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) = 0; \qquad \sum_{k=1}^{3} \mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) = \frac{-1}{4}; \qquad \sum_{k=1}^{4} \mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) = \frac{i}{8} \sum_{j=1}^{4} \cot\left(\pi \frac{\varphi_j}{a_j}\right);$$

$$\sum_{k=1}^{5} \mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) = -\frac{1}{2^4} \sum_{1 \leqslant i < j \leqslant 5} \cot\left(\pi \frac{\varphi_i}{a_i}\right) \cot\left(\pi \frac{\varphi_j}{a_j}\right).$$

Remark 1. For any integer $N \geqslant 0$, by comparing the coefficients of degree $N - (n-1)$ of the right-hand side of (8) we get relations among non-trivial Dedekind–Rademacher sums $S_n(\overrightarrow{A_k}, \overrightarrow{R_k})$ where $1 \leqslant k \leqslant n$ and $\sum_{1 \leqslant j \neq k \leqslant n} r_j = N$, $r_j \geqslant 1$, together with the symmetry between $S_n(\overrightarrow{A_k}, \overrightarrow{R_k})$ for various $\overrightarrow{A_k}$ and $\overrightarrow{R_k}$, let us express all the sums $S_n(\overrightarrow{A_k}, \overrightarrow{R_k})$.

2. Bernoulli functions and proof of main results

We use throughout the notation: $\mathrm{e}(z) = e^{2\pi i z}$ $(z \in \mathbb{C})$.

2.1. Bernoulli functions: An overview

We will need:

- The additive distribution formula satisfied by Bernoulli functions, which is proved by Raabe in [12]

$$\sum_{t=0}^{a-1} \overline{B}_m\left(x + \frac{t}{a}\right) = a^{1-m} \overline{B}_m(ax) \quad \left(a \in \mathbb{N}^*,\ x \in \mathbb{R}\right); \tag{9}$$

- The Fourier expansion formula of $\overline{B}_m(x)$ [6]

$$\overline{B}_m(x) = -\frac{m!}{(2\pi i)^m} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\mathrm{e}(kx)}{k^m} \quad \left(m \in \mathbb{N}^*,\ x \in \mathbb{R}\right). \tag{10}$$

- The generating function of Bernoulli functions is given by [6]: $\forall \phi \in \mathbb{R} \setminus \mathbb{Z}$,

$$F(z, \phi) := \sum_{m=0}^{+\infty} \frac{\overline{B}_m(z)}{m!} (2\pi i \phi)^{m-1} = \begin{cases} \frac{1}{2i} \cot(\pi \phi) & \text{if } z \in \mathbb{Z}, \\ \frac{\mathrm{e}(\{z\}\phi)}{\mathrm{e}(\phi) - 1} & \text{if } z \notin \mathbb{Z}. \end{cases} \tag{11}$$

2.2. Proof of Theorem 1

We start by proving the following two lemmas

Lemma 1.

$$\sum_{k=1}^{n} \mathfrak{S}_n(\overrightarrow{A_k}, \overrightarrow{\Phi_k}) = \sum_{\substack{(t_1, \ldots, t_n) \in \mathbb{N}^n \\ 0 \leqslant t_i \leqslant a_i - 1}} \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j \neq k}}^{n} F\left(\frac{t_k}{a_k} - \frac{t_j}{a_j}, \phi_j\right).$$

Proof. Using the distribution property (9) of $\overline{B}_m$, we can rewrite the definition of $S_n(\overrightarrow{A_k}, \overrightarrow{R_k})$ as

$$S_n(\overrightarrow{A_k}, \overrightarrow{R_k}) \prod_{\substack{j=1 \\ j \neq k}}^{n} a_j^{1-r_j} = \sum_{t_k=0}^{a_k-1} \prod_{\substack{j=1 \\ j \neq k}}^{n} \overline{B}_{r_j}\left(\frac{a_j t_k}{a_k}\right) a_j^{1-r_j}$$

and thus

$$\mathfrak{S}_n(\overrightarrow{A_k},\overrightarrow{\Phi_k})=\sum_{\overrightarrow{R_k}\in\mathbb{N}^{n-1}} S_n(\overrightarrow{A_k},\overrightarrow{R_k})\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{r_j!}\left(\frac{2\pi i\phi_j}{a_j}\right)^{r_j-1}=\sum_{\substack{0\leqslant t_i\leqslant a_i-1\\ 1\leqslant i\leqslant n}}\sum_{\overrightarrow{R_k}\in\mathbb{N}^{n-1}}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{r_j!}\bar{B}_{r_j}\left(\frac{t_k}{a_k}-\frac{t_j}{a_j}\right)(2\pi i\phi_j)^{r_j-1}$$

$$=\sum_{\substack{0\leqslant t_i\leqslant a_i-1\\ 1\leqslant i\leqslant n}}\prod_{\substack{j=1\\ j\neq k}}^{n}\sum_{r_j=0}^{\infty}\frac{1}{r_j!}\bar{B}_{r_j}\left(\frac{t_k}{a_k}-\frac{t_j}{a_j}\right)(2\pi i\phi_j)^{r_j-1}=\sum_{\substack{0\leqslant t_i\leqslant a_i-1\\ 1\leqslant i\leqslant n}}\prod_{\substack{j=1\\ j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right)$$

from which the lemma follows. □

Lemma 2. *Let $(u_1,\dots,u_n)\in\mathbb{R}^n$ such that for all distinct $k,j\in\{1,\dots,n\}$, $u_k-u_j\notin\mathbb{Z}$. Then we have*

$$\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F(u_k-u_j,\phi_j)=0.$$

Proof. The Fourier expansions (10) gives us the identity

$$F(z,\phi)=\frac{1}{2\pi i}\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}(kz)}{\phi-k}=\frac{1}{2\pi i}\sum_{\lambda=\phi\,(\mathrm{mod}\,1)}\frac{\mathrm{e}((\phi-\lambda)z)}{\lambda}.$$

Then

$$\prod_{\substack{j=1\\ j\neq k}}^{n}F(u_k-u_j,\phi_j)=\frac{1}{(2\pi i)^{n-1}}\sum_{\substack{(\lambda_1,\dots,\check{\lambda}_k,\dots,\lambda_n)\equiv(\phi_1,\dots,\check{\phi}_k,\dots,\phi_n)\\ (\mathrm{mod}\ \mathbb{Z}^{n-1})}}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{\mathrm{e}((u_k-u_j)(\phi_j-\lambda_j))}{\lambda_j}.$$

Now, $\sum_{j=1}^{n}\phi_j=0$, then by setting $\lambda_n=-\sum_{j=1}^{n-1}\lambda_j$, one has $=-\sum_{j=1}^{n}u_j(\phi_j-\lambda_j)$ which is independent of k. On the other hand, we know that $\sum_{k=1}^{n}\prod_{j=1,\,j\neq k}^{n}\frac{1}{\lambda_j}=\sum_{k=1}^{n}\lambda_k\prod_{j=1}^{n}\frac{1}{\lambda_j}=0$. We infer therefore

$$\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F(u_k-u_j,\phi_j)=\frac{1}{(2\pi i)^{n-1}}\sum_{k=1}^{n}\sum_{\substack{(\lambda_1,\dots,\check{\lambda}_k,\dots,\lambda_n)\equiv(\phi_1,\dots,\check{\phi}_k,\dots,\phi_n)\\ (\mathrm{mod}\ \mathbb{Z}^{n-1})}}\frac{\mathrm{e}(-\sum_{j=1}^{n}u_j(\phi_j-\lambda_j))}{\prod_{j=1,\,j\neq k}^{n}\lambda_j}=0.$$

This proves Lemma 2. □

Now we compute the sum

$$\sum_{\substack{(t_1,\dots,t_n)\in\mathbb{N}^n\\ 0\leqslant t_i\leqslant a_i-1}}\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right).\tag{12}$$

Thanks to Lemma 2, we know that the contribution in the sum (12) coming from the $\frac{t_k}{a_k}-\frac{t_j}{a_j}\notin\mathbb{Z}$ for all distinct $k,j\in\{1,\dots,n\}$ and all $t_k,t_j\in\mathbb{Z}$, is zero. Let $(a_1,\dots,a_n)$ and $(t_1,\dots,t_n)$ be fixed. We set

$$\sum=\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right).$$

Our aim here is to calculate this quantity $\sum$. We assume first that $(t_1,\dots,t_n)$ is of the form:

$$(t_1,\dots,t_k)=(0,\dots,0,t_{j_0+1},\dots,t_n)\in\prod_{j=1}^{j_0}\{0\}\times\prod_{j=j_0+1}^{n}\,]0,a_j-1]\cap\mathbb{N}.$$

We use the usual convention that a sum over the empty set is 0 and a product over the empty set is 1.

We will observe easily that the general case follows without difficulty. For simplicity, in our computation we can assume that: $\frac{t_j}{a_j}\leqslant\frac{t_{j+1}}{a_{j+1}}$ $(j_0<j<n)$.

We decompose the sum $\sum$ in two sums $\sum_1 + \sum_2$ as $k \leqslant j_0$ or $k > j_0$. Let us put $h_j = t_j/a_j$ $(j = 1, \ldots, n)$. On the one hand, we have

$$\sum_1 := \sum_{k=1}^{j_0} \prod_{\substack{j=1\\ j\neq k}}^{n} F(h_k - h_j, \phi_j) = \prod_{j\leqslant j_0} F(0,\phi_j) \sum_{j\leqslant j_0} \frac{1}{F(0,\phi_j)} \prod_{j>j_0} \frac{\mathrm{e}((1-h_j)\phi_j)}{\mathrm{e}(\phi_j)-1}$$

$$= \frac{\mathrm{e}(-\sum_{j=1}^n h_j\phi_j)}{\prod_{j=1}^n(\mathrm{e}(\phi_j)-1)} \prod_{j\leqslant j_0} \frac{1}{2}\big(1+\mathrm{e}(-\phi_j)\big) \sum_{j\leqslant j_0} \frac{1}{F(0,\phi_j)}.$$

On the other hand, we have

$$\sum_2 = \sum_{k>j_0} \prod_{\substack{j=1\\ j\neq k}}^{n} F(h_k - h_j, \phi_j) = \sum_{k>j_0} \frac{\mathrm{e}(\{h_k - h_j\}\phi_j)}{\prod_{j\neq k}(\mathrm{e}(\phi_j)-1)}$$

$$= \frac{1}{\prod_{j=1}^n(\mathrm{e}(\phi_j)-1)} \sum_{k>j_0} \big(\mathrm{e}(\phi_k)-1\big)\mathrm{e}\bigg(\sum_{j\leqslant k}(h_k-h_j)\phi_j\bigg)\mathrm{e}\bigg(\sum_{j>k}(1-h_j+h_k)\phi_j\bigg)$$

$$= \frac{\mathrm{e}(-\sum_{j=1}^n h_j\phi_j)}{\prod_{j=1}^n(\mathrm{e}(\phi_j)-1)} \bigg(\mathrm{e}\bigg(\sum_{j>j_0}\phi_j\bigg)-1\bigg).$$

We have therefore shown that

$$\sum = \frac{\mathrm{e}(-\sum_{j=1}^n h_j\phi_j)}{\prod_{j=1}^n(\mathrm{e}(\phi_j)-1)} \bigg(\prod_{j\leqslant j_0} \frac{1}{2}\big(\mathrm{e}(-\phi_j)+1\big) \sum_{j\leqslant j_0} \frac{1}{F(0,\phi_j)} + \mathrm{e}\bigg(\sum_{j>j_0}\phi_j\bigg)-1\bigg).$$

For every $\vec{t} = (t_1, \ldots, t_n)$, we set $J(\vec{t}) = \{j\colon\ t_j = 0\}$. By permutation, we can consider the previous case and so

$$\sum = \frac{\mathrm{e}(-\sum_{j=1}^n \frac{t_j}{a_j}\phi_j)}{\prod_{j=1}^n(\mathrm{e}(\phi_j)-1)} \bigg(\mathrm{e}\bigg(-\sum_{j\in J(\vec{t})}\phi_j\bigg) - 1 + \prod_{j\in J(\vec{t})} \frac{1}{2}\big(\mathrm{e}(-\phi_j)+1\big) \sum_{j\in J(\vec{t})} \frac{1}{F(0,\phi_j)}\bigg).$$

We are now ready for the summation over $\vec{t} \in \prod_{j=1}^n [0, a_j - 1] \cap \mathbb{N}$. We see first that for every fixed $J \subset \{1, \ldots, n\}$

$$\sum_{\substack{\vec{t}\\ \{j\colon\ t_j=0\}=J}} \mathrm{e}\bigg(-\sum_{j=1}^n \frac{t_j}{a_j}\phi_j\bigg) = \prod_{j=1}^n \frac{\mathrm{e}(-\frac{\phi_j}{a_j}) - \mathrm{e}(-\phi_j)}{1-\mathrm{e}(-\frac{\phi_j}{a_j})} \prod_{j\in J} \frac{1-\mathrm{e}(-\frac{\phi_j}{a_j})}{\mathrm{e}(-\frac{\phi_j}{a_j}) - \mathrm{e}(-\phi_j)}.$$

In other hand, we have

$$\sum_{J\subset\{1,\ldots,n\}} \prod_{j\in J} \frac{1-\mathrm{e}(-\frac{\phi_j}{a_j})}{\mathrm{e}(-\frac{\phi_j}{a_j}) - \mathrm{e}(-\phi_j)} \mathrm{e}(-\phi_j) = \prod_{i=1}^n \bigg(1 + \frac{1-\mathrm{e}(-\frac{\phi_i}{a_i})}{\mathrm{e}((1-\frac{1}{a_i})\phi_i)-1}\bigg) = \prod_{i=1}^n \frac{\mathrm{e}(\phi_i)-1}{\mathrm{e}(\phi_i)-\mathrm{e}(\frac{\phi_i}{a_i})}.$$

Let us now computing the sum

$$S = \sum_J \prod_{j\in J} \frac{1}{2}\big(\mathrm{e}(-\phi_j)+1\big) \frac{1-\mathrm{e}(-\frac{\phi_j}{a_j})}{\mathrm{e}(-\frac{\phi_j}{a_j}) - \mathrm{e}(-\phi_j)} \sum_{j\in J} \frac{1}{F(0,\phi_j)}$$

where

$$\alpha_j = \frac{1}{2}\big(1+\mathrm{e}(-\phi_j)\big) \frac{1-\mathrm{e}(-\frac{\phi_j}{a_j})}{\mathrm{e}(-\frac{\phi_j}{a_j}) - \mathrm{e}(-\phi_j)} \quad \text{and} \quad \beta_j = \frac{1}{F(0,\phi_j)} = 2\frac{\mathrm{e}(\phi_j)-1}{\mathrm{e}(\phi_j)+1}.$$

Set $H(y) = \sum_J \prod_{j\in J} \alpha_j y^{\beta_j} \prod_{i=1}^n (1+\alpha_i y^{\beta_i})$. From $S = H'(1)$, we have

$$S = \prod_{i=1}^n (1+\alpha_i) \sum_{i=1}^n \frac{\alpha_i\beta_i}{1+\alpha_i} = \prod_{j=1}^n \frac{1}{2} \frac{(1+\mathrm{e}(-\frac{\phi_j}{a_j}))\cdot(1-\mathrm{e}(-\phi_j))}{\mathrm{e}(-\frac{\phi_j}{a_j}) - \mathrm{e}(-\phi_j)} \sum_{j=1}^n 2\frac{1-\mathrm{e}(-\frac{\phi_j}{a_j})}{1+\mathrm{e}(-\frac{\phi_j}{a_j})}.$$

By using Lemma 2 and $\sum_{j=1}^{n}\varphi_j=0$, we arrive at

$$\sum_{\substack{0\leqslant t_j\leqslant a_j-1\\1\leqslant j\leqslant n}}\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right)=\frac{1}{2^{n-1}}\sum_{k=1}^{n}\prod_{1\leqslant j\neq k\leqslant n}\frac{\mathrm{e}(\frac{\phi_j}{a_j})+1}{\mathrm{e}(\frac{\phi_j}{a_j})-1}-\frac{\mathrm{e}(\sum_{j=1}^{n}\frac{\phi_j}{a_j})-1}{\prod_{j=1}^{n}\mathrm{e}(\frac{\phi_j}{a_j})-1}$$

$$=-\frac{1}{(2i)^{n-1}}\frac{\sin(\pi\sum_{j=1}^{n}\frac{\varphi_j}{a_j})}{\prod_{j=1}^{n}\sin(\pi\frac{\varphi_j}{a_j})}+\frac{1}{(2i)^{n-1}}\sum_{k=1}^{n}\prod_{1\leqslant j\neq k\leqslant n}\cot\left(\pi\frac{\varphi_j}{a_j}\right).$$

Finally, using Lemma 1, we obtain Theorem 1.

References

[1] M.F. Atiyah, The logarithm of the Dedekind η-function, Math. Ann. 278 (1987) 335–380.
[2] J. Barge, E. Ghys, Cocycles d'Euler et de Maslov, Math. Ann. 294 (2) (1992) 235–265.
[3] M. Beck, S. Robins, Computing the Continuous Discretely Integer-Point Enumeration in Polyhedra, 1st ed., Springer, 2007.
[4] R. Dedekind, Erläuterungen zu den Fragmenten XXVIII, in: B. Riemann's Gesammelte mathematische Werke, 2nd ed., Teubner, Leipzig, 1892, pp. 466–478.
[5] E. Grosswald, H. Rademacher, Dedekind Sums, Carus Mathematical Monographs, The Mathematical Association of America, 1972.
[6] R.R. Hall, J.C. Wilson, D. Zagier, Reciprocity formulae for general Dedekind–Rademacher sums, Acta Arith. 73 (4) (1995) 389–396.
[7] F. Hirzebruch, The signature theorem: Reminiscences and recreation, in: Prospects in Mathematics, in: Ann. of Math. Studies, vol. 70, Princeton University Press, Princeton, 1971, pp. 3–31.
[8] F. Hirzebruch, D. Zagier, The Atiyah–Singer Theorem and Elementary Number Theory, Math. Lecture Series, vol. 3, Publish or Perish Inc., 1974.
[9] R. Kirby, P. Melvin, Dedekind sums, μ-invariants and the signature cocycle, Math. Annalen 299 (1994) 231–267.
[10] C. Meyer, Über einige Anwendungen Dedekindscher Summen, J. Reine Angew. Math. 198 (1957) 143–203.
[11] D. Quillen, Superconnections and the Chern character, Topology 40 (1985) 89–95.
[12] J.L. Raabe, Zurüchführung eineger Summen und bestimmten Integrale auf die Jacob–Bernoullische Funktion, J. Reine Angew. Math. 42 (1895) 348–367.
[13] H. Rademacher, Generalization of the reciprocity formula for Dedekind sums, Duke Math. J. 21 (1954) 391–397.
[14] U. Weselmann, Eisensteinkohomologie und Dedekindsummen für GL_2 über imaginär-quadratischen Zahlenkörpern, J. Reine Angew. Math. 389 (1988) 90–121.
[15] D. Zagier, Higher order Dedekind sums, Math. Ann. 202 (1973) 149–172.