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Geometry

On almost complex structures which are not compatible with symplectic forms ☆

Sur les structures presque complexes qui ne sont pas compatibles avec des formes symplectiques ☆

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ABSTRACT

In this Note we prove that the underlying almost complex structure to a non-Kähler almost Hermitian structure admitting a compatible connection with skew-symmetric torsion cannot be calibrated by a symplectic form even locally.

RÉSUMÉ

Dans cette Note on démontre que la structure presque complexe sous-jacente à une structure presque hermitienne non kälérienne admettant une connexion compatible avec une torsion antisymétrique ne peut pas, même localement, être calibrée par une forme symplectique.

1. Introduction and preliminaries

Let M be an even dimensional (smooth) manifold. An almost complex structure on M is an endomorphism of the tangent bundle to M satisfying $J^2 = -\mathrm{Id}$. Given an almost complex structure J we denote by N the associated Nijenhuis tensor

$$N(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y].$$

In view of the celebrated theorem of Newlander–Nirenberg (see [5]), N measures how J fails to be a genuine complex structure. A symplectic form ω on M is called *compatible* with a given almost complex structure J if J preserves ω and the tensor

$$g(\cdot,\cdot) := \omega(J\,\cdot\,,\cdot)$$

is a Riemannian metric on M. It is well known that any symplectic form ω admits a compatible almost complex structure. The converse is far from being true even locally.

Using notation of [7], we consider the following:

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Definition 1.1. Let J be an almost complex structure on M and let $p \in M$. J satisfies the *local symplectic property* (*l.s.p.*) at p if there exists a symplectic form ω defined in some neighborhood of p which is compatible with J. We say that J satisfies the *l.s.p.* if it satisfies the *l.s.p.* everywhere. Finally, we say that J does not satisfy the *l.s.p.* if it does not satisfy the *l.s.p.* at any point of M.

In dimension 4 any almost complex structure satisfies the local symplectic property (see [8] at pages 175–176 and [6]), while in dimension greater than 4 things work differently. For instance, Bryant proved in [1] that the standard almost complex structure on S^6 does not satisfy the local symplectic property and Tomassini described in [9] some explicit examples of almost complex structures on $\mathbb{R}^{2n}$ which do not satisfy the local symplectic property. Moreover, from [2,4], we have that the almost complex structure associated to a 6-dimensional strictly nearly Kähler structure does not satisfy the local symplectic property.

We recall that an almost Hermitian structure (g, J) is called *nearly Kähler* if the covariant derivative of J with respect to the Levi-Civita connection of g is a skew-symmetric tensor. If further the Nijenhuis tensor of J does not vanish everywhere, then (g, J) is called a *strictly nearly Kähler structure*. Nearly Kähler structures are naturally endowed by a Hermitian connection with skew-symmetric torsion. An affine connection ∇ on an almost Hermitian manifold (M, g, J) is called *Hermitian with skew-symmetric torsion* if ∇ preserves g, J and the tensor

$$\alpha(X, Y, Z) := g\big(T(X, Y), Z\big)$$

is skew-symmetric, where T denotes the torsion of ∇. The aim of this Note is to prove the following:

Theorem 1.2. *Let (M, g, J) be an almost Hermitian manifold admitting a Hermitian connection with skew-symmetric torsion. Let $p \in M$ such that $N_p \neq 0$. Then J does not satisfy the local symplectic property at p.*

2. Proof of the result

Let (M, g, J) be an almost Hermitian manifold. In view of a result of Friedrich and Ivanov (see [3]), (M, g, J) admits a Hermitian connection with skew-symmetric torsion if and only if

$$\gamma(X, Y, Z) := g\big(N(X, Y), Z\big)$$

is skew-symmetric, where N is the Nijenhuis tensor of J. The almost complex structure J induces the canonical splitting $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$. It is well known that

$$N(Z_i, Z_r) = [Z_i, Z_r]^{0,1} \in T^{0,1}M, \qquad N(Z_i, Z_{\bar r}) = 0, \tag{1}$$

for every $Z_i, Z_r \in T^{1,0}M$, where we set $\overline{Z}_r = Z_{\bar r}$. This yields the following:

Lemma 2.1. *Let (g, J) be an almost Hermitian structure, then*

$$g\big(N(X, Y), Z\big) = g\big(N\big(X^{1,0}, Y^{1,0}\big), Z^{1,0}\big) + g\big(N\big(X^{0,1}, Y^{0,1}\big), Z^{0,1}\big)$$

for every $X, Y, Z \in TM \otimes \mathbb{C}$.

Lemma 2.1 implies that if γ is skew-symmetric, then its complex extension defines a $(3, 0)$-form $\widetilde{\gamma}$ on M, after identifying $T^{1,0}M$ with TM.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Since the result is local, we may assume that M is $\mathbb{R}^{2n}$ and that $p = 0$. Using (1) we get that the $(3, 0)$-form γ associated to (g, J) can be written in terms of bracket as

$$\gamma(Z_1, Z_2, Z_3) = g\big([Z_1, Z_2], Z_3\big)$$

for $Z_1, Z_2, Z_3 \in T^{1,0}M$. Assume that there exists a J-compatible symplectic form ω defined in some neighborhood U of p and let h be the associated almost Kähler metric. We may assume that ω is the standard symplectic form on U. Let A be a (constant) matrix such that $h_p(A\cdot, A\cdot) = g_p(\cdot,\cdot)$. Then $g'(\cdot,\cdot) = h(A\cdot, A\cdot)$ is a metric near p such that $g'_p = g_p$. Now g' is compatible with the almost complex structure $J' = A^{-1}JA$ and $\omega' = g'(J'\cdot\,, \cdot)$ is a non-degenerate 2-form. Since the components of A are constant, ω' is closed and the pair (g', J') is an almost Kähler structure near p. Let $\{Z_r\}$ be a (local) frame of type $(1, 0)$ with respect to J'; then $\{AZ_r\}$ is a frame of type $(1, 0)$ with respect to J near p. Writing $AZ_r = A_r^s Z_s + A_r^{\bar k} Z_{\bar k}$ and using Lemma 2.1 we have at p

$$\begin{aligned}\gamma_p(AZ_r, AZ_l, AZ_i) &= g_p\big(N_p(AZ_r, AZ_l), AZ_i\big)\\ &= g_p\big(N_p\big(A_r^s Z_s, A_l^t Z_t\big), A_i^u Z_u\big) + g_p\big(N_p\big(A_r^{\bar d} Z_{\bar d}, A_l^{\bar o} Z_{\bar o}\big), A_i^{\bar q} Z_{\bar q}\big)\\ &= A_r^s A_l^t A_i^u g'_p\big([Z_s, Z_t]_p, Z_u\big) + A_r^{\bar d} A_l^{\bar o} A_i^{\bar q} g'_p\big([Z_{\bar d}, Z_{\bar o}]_p, Z_{\bar q}\big),\end{aligned}$$

i.e.

$$\gamma_p(AZ_r, AZ_l, AZ_i) = A_r^s A_l^t A_s^u \gamma'_p(Z_s, Z_t, Z_u) + A_r^{\bar d} A_l^{\bar o} A_i^{\bar q} \gamma'_p(Z_{\bar d}, Z_{\bar o}, Z_{\bar q}) \tag{2}$$

where N' is the Nijenhuis tensor of J' and $\gamma'(X, Y, Z) = g'(N'(X, Y), Z)$. Since g' is an almost Kähler metric, γ' satisfies

$$\gamma'(X, Y, Z) + \gamma'(Z, X, Y) + \gamma'(Y, Z, X) = 0.$$

Hence (2) implies that γ at p satisfies

$$\gamma_p(X, Y, Z) + \gamma_p(Z, X, Y) + \gamma_p(Y, Z, X) = 0.$$

Since γ_p is skew-symmetric, this last equation readily implies $N_p = 0$, which is a contradiction. □

Remark 2.2. Note that, as was just observed in dimension 6 in [4], the proof of the above theorem also shows that does not exist a J-compatible almost Hermitian metric g' defined in a neighborhood of p whose fundamental form ω satisfies $(d\omega)^{3,0} = 0$.

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