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Combinatorics/Group Theory

Commutator subgroups of the power subgroups of Hecke groups $H(\lambda_q)$ II

Sous-groupes commutateur de puissance sous-groupes $H(\lambda_q)$ II

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A R T I C L E   I N F O

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A B S T R A C T

Let $q \geqslant 3$ be an odd number and let $H(\lambda_q)$ be the Hecke group associated to q. Let m be a positive integer and let $H^m(\lambda_q)$ be the m-th power subgroup of $H(\lambda_q)$. In this work, we study the commutator subgroups of the power subgroups $H^m(\lambda_q)$ of $H(\lambda_q)$.

R É S U M É

Soit $q \geqslant 3$ un nombre impair et soit $H(\lambda_q)$ le groupe de Hecke associé à q. Soit m un entier positif et soit $H^m(\lambda_q)$ le sous-groupe des puissances m-ièmes de $H(\lambda_q)$. Dans ce travail, nous étudions les sous-groupes commutateurs des puissances sous-groupes $H^m(\lambda_q)$ de $H(\lambda_q)$.

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1. Introduction

In [4], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+\lambda},$$

where λ is a fixed positive real number.

E. Hecke showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, for $q \geqslant 3$ integer, or $\lambda \geqslant 2$. These groups have come to be known as the *Hecke groups*. We consider the former case $q \geqslant 3$ integer and we denote it by $H(\lambda_q)$. Then the Hecke group $H(\lambda_q)$ is the discrete group generated by T and S, and it has a presentation, see [3]:

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_q.$$

Also $H(\lambda_q)$ has the signature $(0; 2, q, \infty)$, that is they are infinite triangle groups. The first several of these groups are $H(\lambda_3) = \Gamma = PSL(2, \mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$.

Let m be a positive integer. Let us define $H^m(\lambda_q)$ to be the subgroup generated by the m-th powers of all elements of $H(\lambda_q)$. The subgroup $H^m(\lambda_q)$ is called the m-th *power subgroup* of $H(\lambda_q)$. As fully invariant subgroups, they are normal in $H(\lambda_q)$.

The power subgroups of the Hecke groups $H(\lambda_q)$ have been studied and classified in [1,5,2]. For $q \geqslant 3$ odd integer and for m positive integer, they proved the following results in [2].

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i) If $(m,2)=1$ and $(m,q)=1$, then the normal subgroup $H^m(\lambda_q)$ is $H(\lambda_q)$, i.e.,

$$H^m(\lambda_q)=H(\lambda_q).$$

ii) If $(m,2)=1$ and $(m,q)=1$, then the normal subgroup $H^m(\lambda_q)$ is the free product of two finite cyclic groups of order q, i.e.,

$$H^m(\lambda_q)=\langle S, TST \mid S^q=(TST)^q=I\rangle \cong \mathbb{Z}_q * \mathbb{Z}_q, \tag{1}$$

and the signature of $H^m(\lambda_q)$ is $(0;q^{(2)},\infty)$.

iii) If $(m,2)=1$ and $(m,q)=d$, then the normal subgroup $H^m(\lambda_q)$ is the free product of d finite cyclic groups of order two and the finite cyclic group of order q/d, i.e.,

$$H^m(\lambda_q)=\langle T\rangle \star \langle STS^{q-1}\rangle \star \langle S^2TS^{q-2}\rangle \star \cdots \star \langle S^{d-1}TS^{q-d+1}\rangle \star \langle S^d\rangle, \tag{2}$$

and it has the signature $(0;2^{(d)},q/d,\infty)$.

iv) If $(m,2)=2$ and $(m,q)=d>2$, then we do not know anything about the normal subgroup $H^m(\lambda_q)$. But, if $d=q$, then

$$H^m(\lambda_q)\subset H'(\lambda_q) \tag{3}$$

and the normal subgroup $H^m(\lambda_q)$ is a free group.

In iii) and iv), if $d=q$ then we can restate these cases as the following.

iii*) If $(m,2)=1$ and $(m,q)=q$, then the normal subgroup $H^m(\lambda_q)$ is the free product of q finite cyclic groups of order two, i.e.,

$$H^m(\lambda_q)=\langle T\rangle \star \langle STS^{q-1}\rangle \star \langle S^2TS^{q-2}\rangle \star \cdots \star \langle S^{q-1}TS\rangle.$$

iv*) If $(m,2)=2$ and $(m,q)=q$, then

$$H^m(\lambda_q)\subset H'(\lambda_q) \tag{4}$$

and the normal subgroup $H^m(\lambda_q)$ is a free group.

In [6], according to all the cases m positive integer, Sahin and Koruoğlu obtained the group structures and the signatures of commutator subgroups of the power subgroups $H^m(\lambda_q)$ of the Hecke groups $H(\lambda_q)$, $q\geqslant 3$ a prime.

In this note, we generalize the results given in [6] for $q\geqslant 3$ a prime number to $q\geqslant 3$ an odd number. In fact, there are similar cases between the case of prime q and the case of odd q. In the above cases i), ii), iii*) and iv*), then all results for $q\geqslant 3$ an odd number coincide with the ones for $q\geqslant 3$ a prime number in [6]. In the case $q\geqslant 3$ an odd number, there are only two cases different from $q\geqslant 3$ prime number case in [6]. These are the cases iii) and iv) except for the cases iii*) and iv*). Since we don't know anything about the normal subgroup $H^m(\lambda_q)$ in the case iv) except for the case iv*), we investigate only the case iii) and obtain the commutator subgroups of the power subgroups $H^m(\lambda_q)$ of the Hecke groups $H(\lambda_q)$.

2. Commutator subgroups of the power subgroups of Hecke groups $H(\lambda_q)$

In this section, we study the commutator subgroups of the power subgroups $H^2(\lambda_q)$ and $H^q(\lambda_q)$ of the Hecke groups $H(\lambda_q)$, for $q\geqslant 3$ odd number.

Theorem 2.1. *Let $q\geqslant 3$ be odd number. If $(m,2)=2$ and $(m,q)=1$, then*

i) $|H^m(\lambda_q):(H^m)'(\lambda_q)|=q^2$,
ii) *the group $(H^m)'(\lambda_q)$ is a free group of rank $(q-1)^2$ with basis* $[S,TST]$, $[S,TS^2T]$, $\ldots$, $[S,TS^{q-1}T]$, $[S^2,TST]$, $[S^2,TS^2T]$, $\ldots$, $[S^2,TS^{q-1}T]$, $\ldots$, $[S^{q-1},TST]$, $[S^{q-1},TS^2T]$, $\ldots$, $[S^{q-1},TS^{q-1}T]$,
iii) *the group $(H^m)'(\lambda_q)$ is of index q in $H'(\lambda_q)$,*
(iv) *for $n\geqslant 2$, $|H^m(\lambda_q):(H^m)^{(n)}(\lambda_q)|=\infty$.*

Theorem 2.2. *Let $q\geqslant 3$ be odd number. If $(m,2)=1$ and $(m,q)=d$, then*

i) $|H^m(\lambda_q):(H^m)'(\lambda_q)|=2^d.\frac{q}{d}$,
ii) *the group $(H^m)'(\lambda_q)$ is a free group of rank $1+(q-2)2^{d-1}$,*
iii) *the group $(H^m)'(\lambda_q)$ is of index 2^{d-1} in $H'(\lambda_q)$,*
(iv) *for $n\geqslant 2$, $|H^m(\lambda_q):(H^m)^{(n)}(\lambda_q)|=\infty$.*

Proof. i) From (1.1), let $k_1 = T$, $k_2 = STS^{q-1}$, $k_3 = S^2TS^{q-2}$, $\ldots$, $k_d = S^{d-1}TS^{q-d+1}$, $k_{d+1} = S^d$. The quotient group $H^m(\lambda_q)/(H^m)'(\lambda_q)$ is the group obtained by adding the relation $k_ik_j = k_jk_i$ to the relations of $H^m(\lambda_q)$, for $i \neq j$ and $i, j \in \{1, 2, \ldots, d+1\}$. Then

$$H^m(\lambda_q)/\big(H^m\big)'(\lambda_q) \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{d \text{ times}} \times \mathbb{Z}_{q/d}.$$

Therefore, we obtain $|H^m(\lambda_q) : (H^m)'(\lambda_q)| = 2^d . \frac{q}{d}$.

ii) Now we choose $\Sigma = \{I, k_1, k_2, \ldots, k_d, k_{d+1}, k_{d+1}^2, \ldots, k_{d+1}^{\frac{q}{d}-1}, k_1k_2, k_1k_3, \ldots, k_1k_d, k_1k_{d+1}, k_1k_{d+1}^2, \ldots, k_1k_{d+1}^{\frac{q}{d}-1}, k_2k_3, \ldots,$ $k_2k_d, k_2k_{d+1}, k_2k_{d+1}^2, \ldots, k_2k_{d+1}^{\frac{q}{d}-1}, \ldots, k_dk_{d+1}, k_dk_{d+1}^2, \ldots, k_dk_{d+1}^{\frac{q}{d}-1}, k_1k_2k_3, \ldots, k_1k_2k_d, k_1k_2k_{d+1}, k_1k_2k_{d+1}^2, \ldots, k_1k_2k_{d+1}^{\frac{q}{d}-1}, \ldots,$ $k_1k_dk_{d+1}, k_1k_dk_{d+1}^2, \ldots, k_1k_dk_{d+1}^{\frac{q}{d}-1}, k_2k_3k_4, \ldots, k_2k_dk_{d+1}, k_2k_dk_{d+1}^2, \ldots, k_1k_dk_{d+1}^{\frac{q}{d}-1}, \ldots, k_1k_2\cdots k_dk_{d+1}, k_1k_2\cdots k_dk_{d+1}^2, \ldots,$ $k_1k_2\cdots k_dk_{d+1}^{\frac{q}{d}-1}\}$ as a Schreier transversal for $(H^m)'(\lambda_q)$. According to the Reidemeister–Schreier method, we get the generators of $(H^m)'(\lambda_q)$ as the followings.

There are $\binom{d}{2}$ generators of the form $k_ik_tk_ik_t$ where $i < t$ and $i, t \in \{1, 2, \ldots, d\}$. There are $2 \times \binom{d}{3}$ generators of the form $k_ik_tk_lk_tk_lk_i$, or $k_ik_tk_lk_ik_lk_t$ where $i < t < l$ and $i, t, l \in \{1, 2, \ldots, d\}$. Similarly, there are $(d-1) \times \binom{d}{d}$ generators of the form $k_1k_2\cdots k_dk_1k_dk_{d-1}\cdots k_2$, or $k_1k_2\cdots k_dk_2k_dk_{d-1}\cdots k_3k_1$, or $\ldots$, or $k_1k_2\cdots k_dk_{d-1}k_dk_{d-2}\cdots k_2k_1$.

Also, there are $\binom{d}{1}\binom{\frac{q}{d}-1}{1}$ generators of the form $k_ik_{d+1}^jk_ik_{d+1}^{d-j}$ where $i \in \{1, 2, \ldots, d\}$ and $j \in \{1, 2, \ldots, \frac{q}{d} - 1\}$. There are $2 \times \binom{d}{2}\binom{\frac{q}{d}-1}{1}$ generators of the form $k_ik_tk_{d+1}^jk_tk_{d+1}^{d-j}k_i$, or $k_ik_tk_{d+1}^jk_ik_{d+1}^{d-j}k_t$ where $i, t \in \{1, 2, \ldots, d\}$, $i < t$ and $j \in \{1, 2, \ldots, \frac{q}{d} - 1\}$. Similarly, there are $d \times \binom{d}{d}\binom{\frac{q}{d}-1}{1}$ generators of the form $k_1k_2\cdots k_dk_{d+1}^jk_1k_{d+1}^{d-j}k_dk_{d-1}\cdots k_2$, or $k_1k_2\cdots k_dk_{d+1}^jk_2k_{d+1}^{d-j}k_dk_{d-1}\cdots k_3k_1$, $\ldots$, or $k_1k_2\cdots k_dk_{d+1}^jk_{d-1}k_{d+1}^{d-j}k_dk_{d-2}\cdots k_2k_1$ where $j \in \{1, 2, \ldots, \frac{q}{d} - 1\}$. In fact, there are totally $1 + (q-2)2^{d-1}$ generators from the theorem of Nielsen.

iii) Since $|H(\lambda_q) : H^m(\lambda_q)| = d$, $|H(\lambda_q) : H'(\lambda_q)| = 2q$ and $|H(\lambda_q) : (H^m)'(\lambda_q)| = q.2^d$, we obtain

$$\big|H'(\lambda_q) : \big(H^m\big)'(\lambda_q)\big| = 2^{d-1}.$$

iv) Please see the proof of the Theorem 2.2 iv). □

Also, we obtain the signature of $(H^m)'(\lambda_q)$ as

$$\big((q-2)2^{d-2} - 2^{(d-2).q/d} + 1; \underbrace{\infty, \infty, \ldots, \infty}_{2^{d-1} \text{ times}}\big) = \big((q-2)2^{d-2} - 2^{(d-2).q/d} + 1; \infty^{(2^{d-1})}\big).$$

Remark 2.1. In Theorem 2.2, if $d = q$ then $\frac{q}{d} - 1 = 0$. Then there is no $\binom{d}{1}\binom{\frac{q}{d}-1}{1} + 2 \times \binom{d}{2}\binom{\frac{q}{d}-1}{1} + \cdots + d \times \binom{d}{d}\binom{\frac{q}{d}-1}{1}$ generators given in the last paragraph of the proof of ii). Thus there are only $\binom{d}{2} + 2 \times \binom{d}{3} + \cdots + (d-1) \times \binom{d}{d} = 1 + (d-2)2^{d-1}$ generators of $(H^m)'(\lambda_q)$. In this case, this result coincides with the Theorem 2.2 in [6].

Example 2.1. Let $q = 9$ and $d = 3$. Then $|H^3(\lambda_9) : (H^3)'(\lambda_9)| = 24$. We choose $\Sigma = \{I, k_1, k_2, k_3, k_4, k_4^2, k_1k_2, k_1k_3, k_1k_4, k_1k_4^2,$ $k_2k_3, k_2k_4, k_2k_4^2, k_3k_4, k_3k_4^2, k_1k_2k_3, k_1k_2k_4, k_1k_2k_4^2, k_1k_3k_4, k_1k_3k_4^2, k_2k_3k_4, k_2k_3k_4^2, k_1k_2k_3k_4, k_1k_2k_3k_4^2\}$ as a Schreier transversal for $(H^3)'(\lambda_9)$. Using the Reidemeister–Schreier method, we get the generators of $(H^3)'(\lambda_9)$ as the following. There are 3 generators of the form,

$$k_1k_2k_1k_2, \quad k_1k_3k_1k_3, \quad k_2k_3k_2k_3,$$

2 generators of the form,

$$k_1k_2k_3k_2k_3k_1, \quad k_1k_2k_3k_1k_3k_2,$$

6 generators of the form,

$$k_1k_4k_1k_4^2, \quad k_2k_4k_2k_4^2, \quad k_3k_4k_3k_4^2,$$
$$k_1k_4^2k_1k_4, \quad k_2k_4^2k_2k_4, \quad k_3k_4^2k_3k_4,$$

12 generators of the form,

$$k_1k_2k_4k_1k_4^2k_2, \quad k_1k_2k_4k_2k_4^2k_1, \quad k_1k_2k_4^2k_1k_4k_2, \quad k_1k_2k_4^2k_2k_4k_1,$$
$$k_1k_3k_4k_1k_4^2k_3, \quad k_1k_3k_4k_3k_4^2k_1, \quad k_1k_3k_4^2k_1k_4k_3, \quad k_1k_3k_4^2k_3k_4k_1,$$
$$k_2k_3k_4k_2k_4^2k_3, \quad k_2k_3k_4k_3k_4^2k_2, \quad k_2k_3k_4^2k_2k_4k_3, \quad k_2k_3k_4^2k_3k_4k_2,$$

and 6 generators of the form,

$$k_1k_2k_3k_4k_1k_4^2k_3k_2, \quad k_1k_2k_3k_4^2k_1k_4k_3k_2,$$
$$k_1k_2k_3k_4k_2k_4^2k_3k_1, \quad k_1k_2k_3k_4^2k_2k_4k_3k_1,$$
$$k_1k_2k_3k_4k_3k_4^2k_2k_1, \quad k_1k_2k_3k_4^2k_3k_4k_2k_1.$$

Therefore, the group $(H^3)'(\lambda_9)$ is a free group of rank 29.

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