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Geometry

Killing vector fields of horizontal Liouville type

Champs de vecteurs de Killing de type de Liouville horizontal

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ABSTRACT

On a slit tangent bundle endowed with a Riemannian metric of Sasaki–Finsler type, we introduce two vector fields of horizontal Liouville type and prove that these vector fields are Killing if and only if the base Finsler manifold is of positive constant curvature. In the special case of one of them, we show that if it is Killing vector field then the base manifold is Einstein–Finsler manifold.

RÉSUMÉ

Sur un fibré tangent doté d'une métrique Riemannienne de type Sasaki–Finsler, nous introduisons deux champs de vecteurs de type de Liouville horizontal et nous prouvons que ces champs sont de Killing si et seulement si la variété de Finsler de base possède une courbure constante positive. Dans le cas particulier de l'un d'entre eux, nous montrons que si le champ de vecteurs est de Killing, alors la base est une variété de Finsler–Einstein.

1. Introduction

The geometry of Einstein–Finsler manifolds and Finsler manifolds of constant curvature are some of the fundamental subjects in Finsler geometry [9,10]. Bryant in [3] has constructed interesting Finsler metrics of positive constant curvature on the sphere S^2. Recently, Deng and Hou have proved in [5] that a homogenous Einstein–Randers space with negative Ricci curvature is Riemannian.

On the slit tangent bundle of a Finsler manifold there are lots of very interesting metrics [7], but we restrict to the Riemannian metric of [8]. In this paper, we introduce vector fields ξ and $\widehat{\xi}$ of horizontal Liouville type, which are unit vector fields with respect to G_S and G, respectively. Then we show that ξ is Killing vector field with respect to G on the indicatrix bundle IM if and only if the Finsler manifold (M,F) is of positive constant curvature $\frac{1}{\beta^2}$. Also, we obtain a condition on Riemannian metric G such that this result is hold for $\widehat{\xi}$. In the special case, we prove that if $\widehat{\xi}$ is Killing vector field with respect to G on IM, then (M,F) is an Einstein–Finsler manifold.

2. Horizontal Liouville vector field with respect to Sasaki metric

Let (M,F) be a Finsler manifold, where M is a real n-dimensional smooth manifold and F is the fundamental function of (M,F) [1]. Consider $TM^\circ = TM \setminus \{0\}$ and denote by VTM° the vertical vector bundle over TM°, that is, $VTM^\circ = \ker \pi_*$, where

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π_* is the tangent mapping of the canonical projection $\pi : TM^\circ \to M$. We may think of the Finsler metric $(g = g_{ij}(x, y))$, where we set $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ as a Riemannian metric on VTM°. The canonical nonlinear connection $HTM^\circ = (N_i^j(x, y))$ of (M, F) is given by $N_i^j = \frac{\partial G^j}{\partial y^i}$, where $G^j = \frac{1}{4}g^{jh}(\frac{\partial^2 F^2}{\partial y^h \partial x^k}y^k - \frac{\partial F^2}{\partial x^h})$. Then on any coordinate neighborhood $\mathfrak{u} \subset TM^\circ$ the vector fields $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$, $i = 1, \ldots, n$, form a basis for $\Gamma(HTM^\circ{}_{|\mathfrak{u}})$. By a straightforward calculation, we obtain the following Lie brackets:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R_{ij}^k \frac{\partial}{\partial y^k}, \qquad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = G_{ij}^k \frac{\partial}{\partial y^k}, \tag{1}$$

where $R_{ij}^k = \frac{\delta N_i^k}{\delta x^j} - \frac{\delta N_j^k}{\delta x^i}$ and $G_{ij}^k = \frac{\partial N_i^k}{\partial y^j}$.

Consider now the energy density $t(x, y) = F^2 = g_{ij}(x, y)y^i y^j$ defined by the Finsler metric F and also the smooth function $v : [0, \infty) \to \mathbb{R}$ and real constants α, β such that $\alpha, \beta > 0$ and $\alpha + tv > 0$ for every t. The above conditions assure that the symmetric $(0, 2)$-type tensor field of TM°, $G_{ij} = \frac{1}{\beta}g_{ij} + \frac{v(t)}{\alpha\beta}y_i y_j$ is positive definite. The inverse of this matrix has the entries $H^{kl} = \beta g^{kl} + \omega(t)y^k y^l$, where (g^{kl}) are the components of the inverse of the matrix (g_{ij}) and $\omega(t) = -\frac{v\beta}{\alpha + tv}$. The components H^{kl} define symmetric $(0, 2)$-type tensor field of TM°. It is easy to see that if the matrix (G_{ij}) is positive definite, then matrix H^{kl} is positive definite too. We use also the components H_{ij} of symmetric $(0, 2)$-type tensor field of TM° obtained from the components H^{kl} by "lowering" the indices $H_{ij} = g_{ik}H^{kl}g_{lj} = \beta g_{ij} + \omega y_i y_j$, where $y_i = g_{ik}y^k$. The following Riemannian metric may be considered on TM°:

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G_{ij}, \qquad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = H_{ij}, \qquad G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0. \tag{2}$$

If $\beta = 1$ and $v(t) = 0$, then the above metric gives us the Sasaki–Finsler metric G_S as follows [1]

$$G_S\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G_S\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}, \qquad G_S\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G_S\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0. \tag{3}$$

It is well known that, on the tangent bundle TM, there exists a globally vector field $\mathbf{S} = y^i \frac{\delta}{\delta x^i}$ called the *horizontal Liouville vector field* [1], or *geodesic spray* [4]. We define horizontal Liouville type vector fields $\xi = l^i \frac{\delta}{\delta x^i} = \frac{1}{F}\mathbf{S}$ and $\widehat{\xi} = \frac{\sqrt{\alpha\beta}}{\sqrt{\alpha + vt}} l^i \frac{\delta}{\delta x^i} = \frac{\sqrt{\alpha\beta}}{F\sqrt{\alpha + vt}}\mathbf{S}$, where $l^i = \frac{y^i}{F}$. Clearly, ξ and $\widehat{\xi}$ are unit vector fields with respect to G_S and G, respectively. For this reason, we call them *G_S-unit horizontal Liouville vector field* and *G-unit horizontal Liouville vector field*.

A Finsler metric F on an n-dimensional manifold M is called an Einstein metric if there is a scalar function $K = K(x)$ on M such that $\mathrm{Ric} = (n-1)KF^2$, where $\mathrm{Ric} = R_k^k$, $R_k^i = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$ and G^i are the geodesic coefficients of F [6]. F is said to be of constant flag curvature $K = \lambda$, if $R_k^i = \lambda(F^2\delta_k^i - FF_{y^k}y^i)$ where $F_{y^k} = \frac{\partial F}{\partial y^k}$ [6]. It is known that a Finsler manifold (M, F) is of constant curvature λ if and only if we have $R_{ij} = \lambda F^2 h_{ij}$, where $R_{ij} = g_{ik}R_j^k$, $h_{ij} = g_{ij} - l_i l_j$ and $l_i = \frac{y_i}{F}$ [2].

By using (1), (2) and Koszol formula, we obtain the following:

Lemma 2.1. *The Levi–Civita connection of the Riemannian metric G defined by* (2) *is as follows*

$$\nabla_{\partial_{\bar{i}}}\partial_{\bar{j}} = \beta^2 C_{ij|0}^s \frac{\delta}{\delta x^s} + \left(C_{ij}^s + \frac{\omega'\alpha + \omega'\upsilon t}{\alpha\beta}y_i y_j y^s + \frac{\omega\alpha + \omega\upsilon t}{\alpha\beta}g_{ij}y^s\right)\frac{\partial}{\partial y^s}, \tag{4}$$

$$\nabla_{\delta_i}\partial_{\bar{j}} = \left(C_{ij}^s + My_i y_j y^s + Ng_{ij}y^s + Ay_i\delta_j^s + \frac{\beta}{2}R_{jki}H^{ks}\right)\frac{\delta}{\delta x^s} + F_{ij}^s \frac{\partial}{\partial y^s}, \tag{5}$$

$$\nabla_{\partial_{\bar{i}}}\delta_j = \left(C_{ij}^s + My_i y_j y^s + Ng_{ij}y^s + Ay_j\delta_i^s + \frac{\beta}{2}R_{ikj}H^{ks}\right)\frac{\delta}{\delta x^s} + \left(F_{ij}^s - G_{ij}^s\right)\frac{\partial}{\partial y^s}, \tag{6}$$

$$\nabla_{\delta_i}\delta_j = F_{ij}^s \frac{\delta}{\delta x^s} + \left(-\frac{1}{\beta^2}C_{ij}^s - Dy_j\delta_i^s - Dy_i\delta_j^s - Ly_i y_j y^s + \frac{1}{2}R_{ij}^s\right)\frac{\partial}{\partial y^s}, \tag{7}$$

where $M = \frac{2\upsilon'\beta + 2\upsilon'\omega t + \upsilon\omega}{2\alpha\beta}$, $N = \frac{\upsilon\beta + \omega\upsilon t}{2\alpha\beta}$, $A = \frac{\upsilon}{2\alpha}$, $D = \frac{\upsilon}{2\alpha\beta^2}$, $L = \frac{\alpha\upsilon' + \upsilon\upsilon' t + \upsilon^2}{\alpha^2\beta^2}$, $C_{ij|0}^s = C_{ij|t}^s y^t$ and $C_{ij|t}^s$ is the h-covariant derivative of C_{ij}^s with respect to Cartan connection.

We recall that the Lie derivative of G with respect to $X \in \chi(TM)$ is given by $(\pounds_X G)(Y, Z) = G(\nabla_Y X, Z) + G(Y, \nabla_Z X)$, where $Y, Z \in \chi(TM)$. By using this formula, we deduce the following:

Lemma 2.2. *The Lie derivative of G with respect to $hX = X^i\delta_i$ satisfies the following*

$$(\pounds_{hX}G)(\delta_i,\delta_j) = X^k_{|i}G_{kj} + X^k_{|j}G_{ki}, \tag{8}$$

$$(\pounds_{hX}G)(\partial_{\bar{i}},\partial_{\bar{j}}) = \beta X^k\big(F^s_{ik} - G^s_{ik}\big)g_{sj} + \beta X^k\big(F^s_{jk} - G^s_{jk}\big)g_{si}, \tag{9}$$

$$\begin{aligned}(\pounds_{hX}G)(\delta_i,\partial_{\bar{j}}) = X^k\bigg[&\bigg(\frac{N\alpha + N\upsilon t}{\alpha\beta} - \beta D\bigg)g_{kj}y_i + \bigg(\frac{M\alpha + M\upsilon t + A\upsilon}{\alpha\beta} - 2\omega D - \beta L - \omega Lt\bigg)y_iy_jy_k + \beta R_{jik}\bigg]\\ &+ \partial_{\bar{j}}\big(X^k\big)G_{ik}.\end{aligned} \tag{10}$$

Lemma 2.3. *The G-unit horizontal Liouville vector field $\widehat{\xi}$ is a Killing vector field on the indicatrix bundle IM if and only if*

$$g_{ij}(x,y) - \frac{v't^2 + vt + \alpha}{\alpha\beta}l_il_j - \beta^2R_{ij}(x,y) = 0, \quad \forall(x,y) \in IM. \tag{11}$$

Proof. It is known that $\widehat{\xi}$ is Killing vector field on IM with respect to G if and only if $\pounds_{\widehat{\xi}}G = 0$ [1]. By using Lemma 2.2, we get $(\pounds_{\widehat{\xi}}G)(\delta_i,\partial_{\bar{j}}) = \frac{\sqrt{\alpha\beta}}{\beta F\sqrt{\alpha+vt}}(g_{ij} - \frac{v't^2+vt+\alpha}{\alpha\beta}l_il_j - \beta^2R_{ij})$ and $(\pounds_{\widehat{\xi}}G)(\delta_i,\delta_j) = (\pounds_{\widehat{\xi}}G)(\partial_{\bar{i}},\partial_{\bar{j}}) = 0$. Hence $\widehat{\xi}$ is Killing on IM if and only if $(\pounds_{\widehat{\xi}}G)(\delta_i,\partial_{\bar{j}}) = 0$. □

We discuss on (11) in the following two cases:

Case 1. If $v't^2 + vt + \alpha = 0$, then we derive the linear equation $v' + \frac{1}{t}v = -\frac{\alpha}{t^2}$ with the solution $v = \frac{1}{t}(c - \alpha\ln t)$, where c is a real constant. In this case, we deduce the following theorem:

Theorem 2.4. *Let (M,F) be a Finsler manifold and G be the Riemannian metric defined by (2) with $v = \frac{1}{t}(c - \alpha\ln t)$ on TM°. If the G-unit horizontal Liouville vector field $\widehat{\xi}$ is Killing on the indicatrix bundle IM, then (M,F) is an Einstein–Finsler manifold.*

Proof. Since $v = \frac{1}{t}(c - \alpha\ln t)$, then by using (11) we have $R_{ij}(x,y) = \frac{1}{\beta^2}g_{ij}(x,y)$ for any $(x,y) \in IM$. Now, take a point $(x,y) \in TM^\circ \setminus IM$. Then there exists $a \in (0,\infty) \setminus \{1\}$ such that $F(x,y) = a$. Since F is positive homogenous of degree 1 with respect to y, then we have $F(x,\frac{1}{a}y) = 1$. Hence $(x,\frac{1}{a}y) \in IM$ and by (11), we obtain $R_{ij}(x,\frac{1}{a}y) = \frac{1}{\beta^2}g_{ij}(x,\frac{1}{a}y)$. Taking into account that g_{ij} and R_{ij} are positively homogenous of degree 0 and 2, respectively, we infer that $R_{ij}(x,y) = \frac{1}{\beta^2}(a^2g_{ij}(x,y)) = \frac{1}{\beta^2}F^2(x,y)g_{ij}(x,y)$. Contracting this equation by g^{ij} we get $\mathrm{Ric} = R^i_i = \frac{n}{\beta^2}F^2$. Hence (M,F) is Einstein manifold with constant function $\frac{n}{(n-1)\beta^2}$. □

Case 2. If $v't^2 + vt + \alpha = \alpha\beta$, then we have the linear equation $v' + \frac{1}{t}v = \frac{\alpha(\beta-1)}{t^2}$. It is easy to check that $v = \frac{1}{t}(\alpha(\beta-1)\ln t + c)$ is an answer of this equation. This case gives us the following theorem:

Theorem 2.5. *Let (M,F) be a Finsler manifold and G be the Riemannian metric defined by (2) with $v = \frac{1}{t}(\alpha(\beta-1)\ln t + c)$ on TM°. Then G-unit horizontal Liouville vector field $\widehat{\xi}$ is Killing on the indicatrix bundle IM if and only if the Finsler manifold (M,F) is of positive constant curvature $\frac{1}{\beta^2}$.*

Proof. If (M,F) is of constant curvature $\frac{1}{\beta^2}$, then we have $R_{ij}(x,y) = \frac{1}{\beta^2}F^2(x,y)h_{ij}(x,y)$ for any $(x,y) \in TM^\circ$. From this equation we obtain (11), since $F(x,y) = 1$ on IM. Conversely, suppose $\widehat{\xi}$ is a Killing vector field on IM. Then by using (11) we have $R_{ij}(x,y) = \frac{1}{\beta^2}h_{ij}(x,y)$ for all $(x,y) \in IM$. Similar to proof of Theorem 2.4, we can deduce $R_{ij}(x,y) = \frac{1}{\beta^2}F^2(x,y)h_{ij}(x,y)$ for any $(x,y) \in TM^\circ$. Hence (M,F) is of positive constant curvature $\frac{1}{\beta^2}$. □

Next, we consider G_S-unit horizontal Liouville vector field ξ. By using Lemma 2.2, we get $(\pounds_\xi G)(\delta_i,\delta_j) = (\pounds_\xi G)(\partial_{\bar{i}},\partial_{\bar{j}}) = 0$ and $(\pounds_\xi G)(\delta_i,\partial_{\bar{j}}) = \frac{1}{\beta F}(h_{ij} - \beta^2R_{ij})$. Therefore, we have the following:

Lemma 2.6. *The G_S-unit horizontal Liouville vector field ξ is a Killing vector field on the indicatrix bundle IM, with respect to G, if and only if*

$$h_{ij} = \beta^2R_{ij}(x,y), \quad \forall(x,y) \in IM.$$

By using the above equation, similar to proof of Theorem 2.5, we can deduce the following:

Theorem 2.7. *Let (M, F) be a Finsler manifold and G be the Riemannian metric on TM° defined by (2). Then G_S-unit horizontal Liouville vector field ξ is Killing on the indicatrix bundle IM if and only if the Finsler manifold (M, F) is of positive constant curvature $\frac{1}{\beta^2}$.*

References

[1] A. Bejancu, H.R. Farran, A geometric characterization of Finsler manifolds of constant curvature $K = 1$, Int. J. Math. Math. Sci. 23 (2000) 399–407.
[2] A. Bejancu, H.R. Farran, Finsler geometry and natural foliations on the tangent bundle, Rep. Math. Phys. 58 (2006) 131–146.
[3] R.L. Bryant, Finsler structures on the 2-sphere satisfying $K = 1$, Amer. Math. Soc., 1996, pp. 27–41.
[4] M. Crasmareanu, Liouville and geodesic Ricci soliton, C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1305–1308.
[5] S. Deng, Z. Hou, Homogenous Eintein–Randers spaces of negative Ricci curvature, C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1169–1172.
[6] E. Guo, X. Mo, X. Zhang, The explicit constraction of Einstein Finsler metrics with non-constant flag curvature, SIGMA Symmetry Integrability Geom. Methods Appl. (2009) 1–7.
[7] R. Miron, D. Hrimiuc, H. Shimada, V.S. Sabau, The Geometry of Hamilton and Lagrange Spaces, Fundam. Theor. Phys., vol. 118, Kluwer Academic Publishers, 2001.
[8] E. Peyghan, A. Tayebi, A Kähler structure on Finsler spaces with non-zero constant flag curvature, J. Math. Phys. 51 (2010) 1–11.
[9] S. Vacaru, Finsler and Lagrange geometries in Einstein and string gravity, Int. J. Geom. Methods Mod. Phys. 5 (2008) 473–511.
[10] S. Vacaru, Finsler black holes induced by noncommutative an holonomic distributions in Einstein gravity, Classical Quantum Gravity 27 (2010) 1–19.