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# Variational forms for the inverses of integral logarithmic operators over an interval

## *Formulations variationnelles pour les inverses des opérateurs intégraux logarithmiques définis sur un intervalle*

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ABSTRACT

We present explicit and exact variational formulations for the weakly singular and hypersingular operators over an interval as well as for their corresponding inverses. By decomposing the solutions in symmetric and antisymmetric parts, we precisely characterize the associated Sobolev spaces. Moreover, we are able to define novel Calderón-type identities in each case.


RÉSUMÉ

Nous présentons des formulations variationnelles explicites et exactes pour les opérateurs intégraux faiblement singulier et hyper-singulier définis sur un interval borné ainsi que pour leurs inverses. En décomposant les solutions en parties symétriques et anti-symétriques, nous caractérisons les espaces de Sobolev associés et retrouvons des identités du type Calderón dans chaque cas.


## Version française abrégée

Dans cette Note, nous étudions les solutions de problèmes associés au Laplacien dans $\mathbb{R}^2 \setminus \Gamma_c$, où $\Gamma_c := (-1,1) \times \{0\}$, pour des conditions de Dirichlet, de Neumann et de leurs sauts à travers $\Gamma_c$ – ce qui conduit à résoudre quatre problèmes différents – à partir de leurs représentations intégrales (18) et (24).

Dans une première étape, nous résolvons ces problèmes dans le cas de conditions de Dirichlet et Neumann différentes de chaque côté du segment fini $\Gamma_c$ (voir Propositions 1 et 2). Puis nous décomposons les solutions volumiques en introduisant la moyenne et la différence ou saut de part et d'autre du plan de symétrie (6). Cette décomposition conduit à des propriétés des opérateurs de traces associés (10). Les Corollaires 3 et 4 identifient les opérateurs symétriques et anti-symétriques avec les opérateurs des problèmes avec conditions de Dirichlet et de saut de Dirichlet originaux. Les Propositions 5 et 6 donnent les résultats similaires pour les cas avec conditions de Neumann.

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Nos résultats principaux sont les expressions explicites (voir Propositions 7 et 8) pour les inverses des opérateurs logarithmiques (16). Finalement, la Proposition 11 donne une formulation variationnelle explicite des inverses des opérateurs logarithmiques dans les différents espaces fonctionnels. Ces formulations explicites dans le cas du Laplacien, permettent de construire des préconditionneurs pour des opérateurs plus géneraux.

## 1. Introduction

We analyze the ubiquitous logarithmic singular operators arising when solving screen, crack or interface problems with piecewise constant coefficients in two dimensions via boundary integral equations. In particular, we focus on the associated weakly singular and hypersingular boundary operators as well as on their inverses. Solutions can be constructed in terms of single and double layer potentials after solving a Fredholm integral equation obtained when taking traces of integral representations and imposing given boundary conditions. When the boundary is closed, Calderón identities hold even for Lipschitz boundaries, with their beneficial properties as preconditioners [4], and Dirichlet and Neumann trace spaces are dual to each other.

The situation changes drastically when considering open manifolds. Indeed, Calderón identities break down due to the disappearance of the double layer potential – and its adjoint – and the mapping properties of the boundary operators degenerate. The resulting logarithmic integral equations have received considerable attention in the past [1–3,6]. However, it seems that a complete framework relating: (i) the exact characterization of the occurring functional spaces; (ii) mapping properties of the operators; and, (iii) explicit formulations for the associated inverse operators; has been elusive so far. In this Note, we address all these items under a unifying scheme for the Laplace operator but these results can be extended through perturbations. Proofs can be found in [5] and are based on extensions and combinations of many results derived in Hölder spaces [10], weighted $L^2$-spaces [9] and Tchebychev polynomials [7].

### 1.1. Preliminaries

Let $\mathcal{O} \subseteq \mathbb{R}^d$, with $d = 1, 2$, be open. For $s \in \mathbb{R}$, $H^s(\mathcal{O})$ denotes standard Sobolev spaces [8]. If $s > 0$ and $\mathcal{O}$ is Lipschitz, $\widetilde{H}^s(\mathcal{O})$ denotes the space of functions whose extension by zero over a closed domain $\widetilde{\mathcal{O}}$ belongs to $H^s(\widetilde{\mathcal{O}})$. We make the following identifications:

$$\widetilde{H}^{-1/2}(\mathcal{O}) \equiv \left(H^{1/2}(\mathcal{O})\right)' \quad \text{and} \quad H^{-1/2}(\mathcal{O}) \equiv \left(\widetilde{H}^{1/2}(\mathcal{O})\right)', \tag{1}$$

where if $\mathcal{O} = \widetilde{\mathcal{O}}$, $\widetilde{H}^{\pm 1/2}(\mathcal{O}) \equiv H^{\pm 1/2}(\mathcal{O})$. $\mathbb{S}'(\mathcal{O})$ is the Schwartz space of tempered distributions.

Without loss of generality, introduce the canonic splitting of $\mathbb{R}^2$ into half-planes $\pi_\pm := \{\mathbf{x} \in \mathbb{R}^2\colon\ x_2 \gtrless 0\}$ with interface $\Gamma$ given by the line $x_2 = 0$. The interface is further divided into the open disjoint segments $\Gamma_c := I \times \{0\}$ and $\Gamma_f := \Gamma \setminus \bar{\Gamma}_c$, where $I := (-1, 1)$.

For simplicity, we focus on the Laplace equation with Dirichlet and Neumann conditions over the cut domain $\Omega := \mathbb{R}^2 \setminus \bar{\Gamma}_c$. Since $\Omega$ is unbounded, one works in the weighted Sobolev space $W^{1,-1}(\Omega)$ [11]. Traces along $\Gamma$ for elements in $W^{1,-1}(\Omega)$ lie in the usual $H^{1/2}_{\mathrm{loc}}(\Gamma)$, and their restriction to a bounded $\Gamma_c$ generates the subspace $H^{1/2}(\Gamma_c)$. Lastly, let us introduce the space $\widetilde{H}^{-1/2}_0(\Gamma_c)$ as the subspace of $\widetilde{H}^{-1/2}(\Gamma_c)$-distributions with zero mean value, i.e.

$$\widetilde{H}^{-1/2}_0(\Gamma_c) = \left\{\varphi \in \widetilde{H}^{-1/2}(\Gamma_c)\colon\ \langle \varphi, 1\rangle_{\Gamma_c} = 0\right\}. \tag{2}$$

Lastly, trace operators over $\Gamma_c$, denoted by $\gamma_c^\pm$, are taken from within $\pi_\pm$. The symbol $[\cdot]$ represents the jump operator and $\partial_n = \mathbf{n} \cdot \nabla$ with $\mathbf{n}$ pointing outwards for closed boundaries. In the case of $\Gamma$, being non-orientable as a boundary manifold, we assume $\mathbf{n}$ pointing along the positive $x_2$-axis.

### 1.2. Dirichlet problems

We consider the Laplace problem with two different Dirichlet conditions $g^\pm$ from above and below on $\Gamma_c$. This boundary data lies in the Hilbert space:

$$\begin{aligned} &\mathbb{X} := \left\{\mathbf{g} = \left(g^+, g^-\right) \in H^{1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c) : g^+ - g^- \in \widetilde{H}^{1/2}(\Gamma_c)\right\} \quad \text{with norm} \\ &\|\mathbf{g}\|^2_{\mathbb{X}} := \left\|g^+\right\|^2_{H^{1/2}(\Gamma_c)} + \left\|g^-\right\|^2_{H^{1/2}(\Gamma_c)} + \left\|g^+ - g^-\right\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}. \end{aligned} \tag{3}$$

Equivalently, we define the Hilbert space for Neumann data:

$$\mathbb{Y} := \left\{\boldsymbol{\varphi} = \left(\varphi^+, \varphi^-\right) \in H^{-1/2}(\Gamma_c) \times H^{-1/2}(\Gamma_c)\colon\ \varphi^+ - \varphi^- \in \widetilde{H}^{-1/2}_0(\Gamma_c)\right\} \tag{4}$$

with similar norm. The Dirichlet problem we consider is: for $\mathbf{g} \in \mathbb{X}$, find $u \in W^{1,-1}(\Omega)$ such that:

$$\begin{cases} -\Delta u = 0, & \mathbf{x} \in \Omega, \\ \begin{pmatrix} \gamma_c^+ \\ \gamma_c^- \end{pmatrix} u = \mathbf{g}, & \mathbf{x} \in \Gamma_c. \end{cases} \tag{5}$$

**Proposition 1.** *If* $\mathbf{g} \in \mathbb{X}$*, then problem* (5) *has a unique solution in* $W^{1,-1}(\Omega)$.

The problem can be split in the following way. To any function $u$ in $W^{1,-1}(\Omega)$, one associates restrictions $u^{\pm}$ on $\pi_{\pm}$ belonging to $W^{1,-1}(\pi_{\pm})$. Denote by $\check{u}^{\pm} \in W^{1,-1}(\mathbb{R}^d)$ the mirror reflection of $u^{\pm}$ over $\pi_{\mp}$. Then, one can introduce average and jump solutions as

$$\begin{cases} u_{\mathrm{avg}} := \dfrac{1}{2}\left(\check{u}^+ + \check{u}^-\right), \\ u_{\mathrm{jmp}} := \dfrac{1}{2}\left(\check{u}^+ - \check{u}^-\right), \end{cases} \quad \text{associated to the data} \quad \begin{cases} g_{\mathrm{avg}} := \dfrac{1}{2}\left(g^+ + g^-\right), \\ g_{\mathrm{jmp}} := \dfrac{1}{2}\left(g^+ - g^-\right). \end{cases} \tag{6}$$

Normal traces can be similarly decomposed. Due to the convened orientation of the normal on $\Gamma_c$, they take the form:

$$\begin{cases} \gamma_c \partial_n u_{\mathrm{avg}} = \dfrac{1}{2}\mathbf{n} \cdot \nabla\left(\check{u}^+ - \check{u}^-\right), \\ \gamma_c \partial_n u_{\mathrm{jmp}} = \dfrac{1}{2}\mathbf{n} \cdot \nabla\left(\check{u}^+ + \check{u}^-\right), \end{cases} \quad \text{corresponding to} \quad \begin{cases} u_{\mathrm{avg}}, \\ u_{\mathrm{jmp}}, \end{cases} \tag{7}$$

respectively, and we have the associated Green' formula:

$$(\nabla u, \nabla v)_{\Omega} = \langle \gamma_c \partial_n u_{\mathrm{avg}}, \gamma_c v_{\mathrm{avg}} \rangle_{H^{1/2}(\Gamma_c)} + \langle \gamma_c \partial_n u_{\mathrm{jmp}}, \gamma_c v_{\mathrm{jmp}} \rangle_{\widetilde{H}^{1/2}(\Gamma_c)}, \tag{8}$$

since $(\nabla u_{\mathrm{avg}}, \nabla v_{\mathrm{jmp}})_{\Omega} = 0$, for $v \in W^{1,-1}(\mathbb{R}^2)$ also decomposed in average and jump parts.

**Proposition 2.** *The solution of the Dirichlet isotropic problem* (5)*, is such that its Neumann trace at* $\Gamma_c$ *belongs to the space* $\mathbb{Y}$*. There exists a unique application* $\boldsymbol{\mathcal{D}} : \mathbb{X} \to \mathbb{Y}$ *relating Dirichlet traces to Neumann traces* (*Dirichlet-to-Neumann map or DtN*)*. Moreover, the energy inequality holds*

$$\langle \boldsymbol{\mathcal{D}} \mathbf{g}, \mathbf{g} \rangle_{\Gamma_c} \geqslant C \|\mathbf{g}\|_{\mathbb{X}}^2, \tag{9}$$

*for* $\mathbf{g}$ *in* $\mathbb{X}$ *and where the vector duality product is given by*:

$$\langle \boldsymbol{\mathcal{D}} \mathbf{g}, \mathbf{g} \rangle_{\Gamma_c} = \langle \boldsymbol{\mathcal{D}} \mathbf{g}_{\mathrm{avg}}, \mathbf{g}_{\mathrm{avg}} \rangle_{H^{1/2}(\Gamma_c)} + \langle \boldsymbol{\mathcal{D}} \mathbf{g}_{\mathrm{jmp}}, \mathbf{g}_{\mathrm{jmp}} \rangle_{\widetilde{H}^{1/2}(\Gamma_c)}. \tag{10}$$

**Corollary 3.** *If* $g^{\pm} =: g \in H^{1/2}(\Gamma_c) \setminus \mathbb{C}$*, the corresponding solution of* (5) *in* $\Omega$ *is symmetric with respect to* $\Gamma$*. Moreover, there exists a unique DtN operator* $\mathcal{D}_s : H^{1/2}(\Gamma_c) \setminus \mathbb{C} \to \widetilde{H}_0^{-1/2}(\Gamma_c)$*, and the energy inequality holds*

$$\langle \mathcal{D}_s\, g, g \rangle_{\Gamma_c} \geqslant C \|g\|_{H^{1/2}(\Gamma_c)\setminus\mathbb{C}}^2. \tag{11}$$

**Corollary 4.** *If* $g^{\pm} = \pm g \in \widetilde{H}^{1/2}(\Gamma_c)$*, the associated solution of* (5) *is antisymmetric with respect to* $\Gamma$*. Furthermore, there exists a unique DtN operator* $\mathcal{D}_{as} : \widetilde{H}^{1/2}(\Gamma_c) \to H^{-1/2}(\Gamma_c)$ *satisfying*

$$\langle \mathcal{D}_{as}\, g, g \rangle_{\Gamma_c} \geqslant C \|g\|_{\widetilde{H}^{1/2}(\Gamma_c)}^2. \tag{12}$$

### *1.3. Neumann problems*

Symmetric and antisymmetric Neumann problems can stated as follows: find $u_s, u_{as} \in W^{1,-1}(\mathbb{R}^2)$ such that

$$\begin{cases} -\Delta u_s = 0, & \mathbf{x} \in \Omega, \\ [\gamma_c \partial_n u_s] = \varphi, & \mathbf{x} \in \Gamma_c, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_{as} = 0, & \mathbf{x} \in \Omega, \\ \gamma_c^{\pm} \partial_n u_{as} = \phi, & \mathbf{x} \in \Gamma_c, \end{cases} \tag{13}$$

for data $\varphi$ in the space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ and $\phi$ in $H^{-1/2}(\Gamma_c)$. We will refer to the inverse maps as Neumann-to-Dirichlet (NtD) maps.

**Proposition 5.** *The symmetric Neumann problem* (13) *has a unique solution in* $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$ *if and only if* $\varphi \in \widetilde{H}_0^{-1/2}(\Gamma_c)$*. Thus, there exists a unique continuous and invertible NtD, denoted by* $\mathcal{N}_s : \widetilde{H}_0^{-1/2}(\Gamma_c) \to H^{1/2}(\Gamma_c)/\mathbb{C}$ *satisfying*

$$\langle \mathcal{N}_s\, \varphi, \varphi \rangle_{\Gamma_c} \geqslant C \|\varphi\|_{\widetilde{H}_0^{-1/2}(\Gamma_c)}^2. \tag{14}$$

*The inverse of this application is the operator* $\mathcal{D}_s$ *defined in Corollary* 3.

**Proposition 6.** *The antisymmetric Neumann problem* (13) *has a unique solution in* $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$ *if and only if* $\phi \in H^{-1/2}(\Gamma_c)$*. Hence, there exists a unique continuous and invertible* $\mathcal{N}_{as}$ *from* $H^{-1/2}(\Gamma_c) \to \widetilde{H}^{1/2}(\Gamma_c)$*, and the energy inequality holds*

$$\langle \mathcal{N}_{as}\phi, \phi\rangle_{\Gamma_c} \geqslant C\|\phi\|^2_{H^{-1/2}(\Gamma_c)}. \tag{15}$$

*The inverse of this application is the operator* $\mathcal{D}_{as}$ *defined in Corollary* 4.

## 2. Main results

Introduce the following integral logarithmic operators for $x \in I$:

$$\mathcal{L}_1\varphi(x) := \int_I \log\frac{1}{|x-y|}\varphi(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{L}_2\varphi(x) := \int_I \log\left[\frac{M(x,y)}{|x-y|}\right]\varphi(x)\,\mathrm{d}x, \tag{16}$$

where the first one is the standard weakly singular single layer operator and where

$$M(x,y) := \frac{1}{2}\big((y-x)^2 + \big(w(x)+w(y)\big)^2\big), \quad \text{with } w(x) := \sqrt{1-x^2},\ x \in I. \tag{17}$$

Lastly, introduce the subspace $H^{1/2}_*(\Gamma_c)$ of functions in $H^{1/2}(\Gamma_c)$ satisfying $\langle g, w\rangle_{\Gamma_c} = 0$.

### *2.1. Symmetric problem and the weakly singular operator*

In this case, symmetric Dirichlet and Neumann problems are given via the simple layer potential. For the Neumann version, one just simply introduces the data in the potential whereas for the Dirichlet problem one needs to solve: find $\varphi$ such that

$$\mathcal{L}_1\varphi(x) = g(x), \quad x \in I. \tag{18}$$

This integral equation admits an explicit inverse and variational formulations for the equation as well as for its inverse are given in the following proposition:

**Proposition 7.** *The symmetric variational formulation of the integral equation* (18) *in the Hilbert space* $\widetilde{H}_0^{-1/2}(\Gamma_c)$ *is*

$$\langle \mathcal{L}_1\varphi, \varphi^t\rangle_{\Gamma_c} = \langle g, \varphi^t\rangle_{\Gamma_c}, \quad \forall\varphi^t \in \widetilde{H}_0^{-1/2}(\Gamma_c), \tag{19}$$

*and the associated bilinear form is coercive. The associated operator is* $\mathcal{N}_s$ *which is a bijection between* $\widetilde{H}_0^{-1/2}(\Gamma_c)$ *and* $H^{1/2}_*(\Gamma_c)$*. The inverse operator is bijective from* $H^{1/2}_*(\Gamma_c)$ *onto* $\widetilde{H}_0^{-1/2}(\Gamma_c)$ *and is associated to the operator* $\mathcal{D}_s$ *which is symmetric and coercive in the space* $H^{1/2}_*(\Gamma_c)$*. It admits two variational formulations*:

$$\frac{1}{\pi^2}\langle \mathcal{L}_2 g', (g^t)'\rangle_{\Gamma_c} = \langle \varphi, g^t\rangle_{\Gamma_c}, \quad \forall g^t \in H^{1/2}_*(\Gamma_c), \tag{20}$$

*which gives a first norm on the space* $H^{1/2}_*(\Gamma_c)$:

$$\frac{1}{\pi^2}\langle \mathcal{L}_2 g', g'\rangle_{\Gamma_c} \geqslant C\|g\|^2_{H^{1/2}_*(\Gamma_c)}, \quad \forall g \in H^{1/2}_*(\Gamma_c). \tag{21}$$

*The second one is*

$$\frac{1}{2\pi^2}\int_I\int_I \frac{\mathrm{d}^2}{\mathrm{d}x\,\mathrm{d}y}\log\left[\frac{M(x,y)}{|x-y|}\right]\big(g(x)-g(y)\big)\big(g^t(x)-g^t(y)\big)\,\mathrm{d}y\,\mathrm{d}x = \langle\varphi, g^t\rangle_{\Gamma_c}, \tag{22}$$

*for all* $g^t \in H^{1/2}_*(\Gamma_c)$*, and we obtain a second norm on the space* $H^{1/2}_*(\Gamma_c)$ *which is*:

$$\int_{\Gamma_c}\int_{\Gamma_c} \frac{1-xy}{w(x)w(y)}\frac{(g(x)-g(y))^2}{(x-y)^2}\,\mathrm{d}y\,\mathrm{d}x \geqslant C\|g\|^2_{H^{1/2}_*(\Gamma_c)}, \quad \forall g \in H^{1/2}_*(\Gamma_c). \tag{23}$$

Although the Dirichlet problem (5) admits a unique solution for all $g = g^{\pm}$ in $H^{1/2}(\Gamma_c)$, the solution to a constant data, e.g. corresponding to $\gamma_m^{\pm}u = 1$, is such that $\varphi = 0$. Thus, the integral representation (18) cannot describe this constant solution. The exact image by the operator $\mathcal{N}_s$ of the space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ is the subspace $H^{1/2}_*(\Gamma_c)$ which also does not contain the trace of the constant function.

*2.2. Antisymmetric problem and the hypersingular operator*

The solution for the antisymmetric Dirichlet problem is retrieved by direct action of the double layer potential. However, for the Neumann version, one must first solve the hypersingular integral equation for $\alpha$ (the jump of the Dirichlet trace):

$$\varphi(x) = \int\!\!\!\!\!\!- _I \frac{1}{|x-y|^2}\alpha(y)\,\mathrm{d}y, \quad \text{for } x \in I. \tag{24}$$

**Theorem 8.** *A symmetric variational formulation for* (24) *in the Hilbert space $\widetilde{H}^{1/2}(\Gamma_c)$ is given by*

$$\langle \mathcal{L}_1\,\alpha', (\alpha^t)' \rangle_{\Gamma_c} = \langle \varphi, \alpha^t \rangle_{\Gamma_c}, \quad \forall \alpha^t \in \widetilde{H}^{1/2}(\Gamma_c), \tag{25}$$

*which is coercive. The associated operator $\mathcal{D}_{as}$ is a bijection from $\widetilde{H}^{1/2}(\Gamma_c)$ to $H^{-1/2}(\Gamma_c)$. This operator admits an alternative variational formulation*:

$$\int_I\int_I \frac{(\alpha(x)-\alpha(y))(\alpha^t(x)-\alpha^t(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + 2\int_I \frac{\alpha(x)\alpha^t(x)}{1-x^2}\,\mathrm{d}x = \langle \varphi, \alpha^t \rangle_{\Gamma_c}, \tag{26}$$

*for all $\alpha^t \in \widetilde{H}^{1/2}(\Gamma_c)$, and the next expression is a norm on $\widetilde{H}^{1/2}(\Gamma_c)$*:

$$\int_I\int_I \frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + 2\int_I \frac{\alpha(x)^2}{1-x^2}\,\mathrm{d}x \geqslant C\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}, \quad \forall \alpha \in \widetilde{H}^{1/2}(\Gamma_c). \tag{27}$$

*The inverse operator is associated to the operator $\mathcal{N}_{as}^{-1} = \mathcal{D}_{as}$, and it is a bijection of $H^{-1/2}(\Gamma_c)$ onto $\widetilde{H}^{1/2}(\Gamma_c)$, symmetric and coercive in the space $H^{-1/2}(\Gamma_c)$. It admits the following variational formulation*:

$$\frac{1}{\pi^2}\langle \mathcal{L}_2\,\varphi, \varphi^t \rangle_{\Gamma_c} = \langle \alpha, \varphi^t \rangle_{\Gamma_c}, \quad \forall \varphi \in H^{-1/2}(\Gamma_c), \tag{28}$$

*and thus, the following expression is a norm on the space $H^{-1/2}(\Gamma_c)$*:

$$\langle \mathcal{L}_2\,\varphi, \varphi \rangle \geqslant C\|\varphi\|^2_{H^{-1/2}(\Gamma_c)}, \quad \forall \varphi \in H^{-1/2}(\Gamma_c). \tag{29}$$

**Proposition 9.** *The subspace $\widetilde{H}^{1/2}(\Gamma_c)$ is exactly the space of functions $g$ in the space $H_*^{1/2}(\Gamma_c)$ such that $w^{-1}g$ is in $L^2(\Gamma_c)$. The space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ is exactly the subspace of $H^{-1/2}(\Gamma_c)$, orthogonal in the duality product with $\widetilde{H}^{1/2}(\Gamma_c)$ to the space of functions $w\varphi$, where $\varphi$ varies in $L^2(\Gamma_c)$.*

*2.3. Calderón-type identities*

Two derivation operators have appeared in the above propositions, one whose domain lies on $\widetilde{H}^{1/2}(\Gamma_c)$ and another acting on $H_*^{1/2}(\Gamma_c)$. Since $\widetilde{H}^{1/2}(\Gamma_c)$ can be extended by zero to be a subspace of $H^{1/2}(\mathbb{R})$ which is a subspace of the distribution space $\mathbb{S}'(\mathbb{R})$, the first derivation operator, denoted by $D$, is defined distributionally. We will denote the second one as $-D^*$ taken in classical sense.

**Proposition 10.** *The derivation operator $D$ is continuous and surjective from the space $\widetilde{H}^{1/2}(\Gamma_c)$ onto $\widetilde{H}_0^{-1/2}(\Gamma_c)$, while the derivation operator $-D^*$ is continuous and surjective from the space $H_*^{1/2}(\Gamma_c)$ onto the space $H^{-1/2}(\Gamma_c)$. Moreover, the operator $D^*$ is the adjoint of the operator $D$ with respect to the duality product in $L^2(\Gamma_c)$.*

Finally, one can prove some properties linking these derivation operators $D$ and $D^*$ and the logarithmic operators previously introduced.

**Proposition 11.** *The operators $\mathcal{L}_1$, $\mathcal{L}_2$, $D$, $D^*$ are linked by the identities*

$$-\mathcal{L}_2 \circ D^* \circ \mathcal{L}_1 \circ D = \mathrm{Id}_{\widetilde{H}^{1/2}(\Gamma_c)}, \qquad -\mathcal{L}_1 \circ D \circ \mathcal{L}_2 \circ D^* = \mathrm{Id}_{H_*^{1/2}(\Gamma_c)},$$
$$-D \circ \mathcal{L}_2 \circ D^* \circ \mathcal{L}_1 = \mathrm{Id}_{\widetilde{H}_0^{-1/2}(\Gamma_c)}, \qquad -D^* \circ \mathcal{L}_2 \circ D \circ \mathcal{L}_1 = \mathrm{Id}_{H^{-1/2}(\Gamma_c)}.$$

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