C. R. Acad. Sci. Paris, Ser. I 349 (2011) 251–254

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical Analysis/Functional Analysis

Some conditions implying normality of operators

Quelques conditions entraînant la normalité d'opérateurs

M.S. Moslehian [a], S.M.S. Nabavi Sales [b,c]

[a] *Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran*
[b] *Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran*
[c] *Tusi Mathematical Research Group (TMRG), Mashhad, Iran*

ARTICLE INFO

Article history:
Received 25 November 2010
Accepted after revision 18 January 2011
Available online 2 February 2011

Presented by the Editorial Board

ABSTRACT

Let $T \in \mathbb{B}(\mathscr{H})$ and $T = U|T|$ be its polar decomposition. We prove that (i) if T is log-hyponormal or p-hyponormal and $U^n = U^*$ for some n, then T is normal; (ii) if the spectrum of U is contained in some open semicircle, then T is normal if and only if so is its Aluthge transform $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$.

RÉSUMÉ

Soit $T \in \mathbb{B}(\mathcal{H})$ et $T = U|T|$ sa décomposition polaire. Nous montrons que : (i) si T est log-hyponormal ou p-hyponormal et $U^n = U^*$ pour un certain n, alors T est normal ; (ii) si le spectre de U est contenu dans un arc de cercle, alors T est normal si et seulement s'il en est de même de son transformé de Aluthge $\widetilde{T} = |T|^{1/2} U |T|^{1/2}$.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity I. A subspace $\mathscr{K} \subseteq \mathscr{H}$ is said to reduce $T \in \mathbb{B}(\mathscr{H})$ if both $T\mathscr{K} \subseteq \mathscr{K}$ and $T^*\mathscr{K} \subseteq \mathscr{K}$ hold. We say that an operator T is p-hyponormal for some $p > 0$ if $(T^*T)^p \geqslant (TT^*)^p$. If $p = 1$, T is said to be hyponormal. Clearly T is hyponormal if and only if $\|T\xi\| \geqslant \|T^*\xi\|$ for any $\xi \in \mathscr{H}$. If T is an invertible operator satisfying $\log(T^*T) \geqslant \log(TT^*)$, then it is called log-hyponormal, see [13].

Let $T = U|T|$ be the polar decomposition of T, where $\ker(U) = \ker(|T|)$ and U^*U is the projection onto $\overline{\operatorname{ran}(|T|)}$. It is known that if T is invertible, then U is unitary and $|T|$ is also invertible. It is easy to see that

$$|T^*|^s = U|T|^s U^* \tag{1}$$

for every nonnegative number s. If T is invertible, then

$$\log|T^*| = U\big(\log|T|\big)U^*. \tag{2}$$

The Aluthge transformation $\widetilde{T}$ of T is defined by $\widetilde{T} := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. This notion was first introduced by Aluthge [1] and is a powerful tool in the operator theory. The reader is referred to [7] for undefined notions and terminology.

E-mail addresses: moslehian@ferdowsi.um.ac.ir, moslehian@ams.org (M.S. Moslehian), sadegh.nabavi@gmail.com (S.M.S. Nabavi Sales).
URL: http://profsite.um.ac.ir/~moslehian/ (M.S. Moslehian).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.01.018

One interesting problem in the operator theory is to investigate some conditions under which certain operators are normal. Several mathematicians have paid attention to this problem, see [1,2,6,8] and references therein. One of interesting articles, which presents some results about this topic is that of Stampfli [11]. He showed, among other things, that for a hyponormal operator A, if A^n is normal for some positive integer n, then A is normal. The problem had already been considered in the case when $n=2$ by Putnam [9]. The results were generalized later to the other classes of operators by a number of authors, for instance, Embry [5], Radjavi and Rosenthal [10] and Duggal [4]. There is another point of view about this issue via spectrum $\mathrm{sp}(\cdot)$. In [11] it is proved that if the spectrum of a hyponormal operator contains only a finite number of limited points or has zero area, then the operator is normal. Using Aluthge transform, this aspect is generalized to p-hyponormal and log-hyponormal operators. In fact, if T is p-hyponormal or log-hyponormal, then $\widetilde{\widetilde{T}}$ is hyponormal [7, Theorem 1.3.4.1 and Theorem 2.3.4.2]. Due to $\mathrm{sp}(A)=\mathrm{sp}(\widetilde{\widetilde{A}})$ [2, Corollary 2.3], $\widetilde{\widetilde{A}}$ is normal. Now the result is concluded from the fact that $\widetilde{\widetilde{A}}$ is normal if and only if so is A [14, Lemma 3]. There are some applications of the subject in other areas of the operator theory that was a motivation for our work, see [8].

In this paper we present some new conditions under which certain operators are normal. We also use a Fuglede–Putnam commutativity type theorem to show that an invertible operator $T=U|T|$, where $\mathrm{sp}(U)$ is contained in an open semicircle, is normal if and only if so is $\widetilde{T}$.

2. Main results

We start this section with one of our main results:

Theorem 2.1. *Let $T\in\mathbb{B}(\mathscr{H})$ be* log*-hyponormal or p-hyponormal and $T=U|T|$ be the polar decomposition of T such that $U^{n_0}=U^*$ for some positive integer n_0. Then T is normal.*

Proof. Assume that T is p-hyponormal for some $p>0$. Hence $|T|^{2p}\geqslant|T^*|^{2p}=U|T|^{2p}U^*$ by (1). By multiplying both sides of this inequality by U and U^* we have $U|T|^{2p}U^*\geqslant U^2|T|^{2p}U^{2*}$ whence $|T|^{2p}\geqslant U|T|^{2p}U^*\geqslant U^2|T|^{2p}U^{2*}$. By repeating this process, we reach the following sequence of operator inequalities:

$$|T|^{2p}\geqslant|T^*|^{2p}=U|T|^{2p}U^*\geqslant U^2|T|^{2p}U^{2*}\geqslant\cdots\geqslant U^{n_0+1}|T|^{2p}U^{(n_0+1)*}\geqslant\cdots. \tag{3}$$

Because of $U^{n_0}=U^*$ we can observe that $U^{n_0+1}=U^*U=U^{(n_0+1)*}$ is the projection onto $\overline{\mathrm{ran}(|T|)}$. Hence $U^{n_0+1}|T|^{2p}U^{(n_0+1)*}=|T|^{2p}$, from which and inequalities (3) we obtain $|T|^{2p}=|T^*|^{2p}$. Hence $|T|^2=|T^*|^2$, i.e., T is normal as desired.

In the case that T is a log-hyponormal operator inequalities (3) are replaced by the inequalities

$$\log|T|\geqslant\log|T^*|=U\big(\log|T|\big)U^*\geqslant U^2\big(\log|T|\big)U^{2*}\geqslant\cdots\geqslant U^{n_0+1}\big(\log|T|\big)U^{(n_0+1)*}\geqslant\cdots$$

and the rest of the proof is similar to argument above. □

We will need the following lemma in the sequel. One can easily prove it by using the fact that $\log(cT)=(\log c)I+\log T$.

Lemma 2.2. *If T and S are two invertible positive operators such that* $\log T\geqslant\log S$ *and c is a positive number, then* $\log(cT)\geqslant\log(cS)$.

Theorem 2.3. *Let $T\in\mathbb{B}(\mathscr{H})$ be* log*-hyponormal or p-hyponormal and $T=U|T|$ be the polar decomposition of T such that $U^{*n}\to I$ or $U^n\to I$ as $n\to\infty$, where limits are taken in the strong operator topology. Then T is normal.*

Proof. We assume that $U^{*n}\xi\to\xi$ as $n\to\infty$ for all $\xi\in\mathscr{H}$. In the case $U^n\to I$ in the strong operator topology a similar argument can be used. Let T be p-hyponormal and $\xi\in\mathscr{H}$. It follows from (3) that

$$\big\||T|^p\xi\big\|\geqslant\big\||T^*|^p\xi\big\|=\big\||T|^pU^*\xi\big\|\geqslant\big\||T|^pU^{2*}\xi\big\|\geqslant\cdots\geqslant\big\||T|^pU^{n*}\xi\big\|\geqslant\cdots. \tag{4}$$

Since

$$\Big|\big\||T|^pU^{*n}\xi\big\|-\big\||T|^p\xi\big\|\Big|\leqslant\big\||T|^pU^{*n}\xi-|T|^p\xi\big\|\leqslant\big\||T|^p\big\|\big\|U^{*n}\xi-\xi\big\|\to 0$$

as $n\to\infty$, we have $\||T|^pU^{*n}\xi\|\to\||T|^p\xi\|$ as $n\to\infty$. Hence, by (4) we get $\||T|^p\xi\|^2=\||T^*|^p\xi\|^2$, so $|T|^{2p}=|T^*|^{2p}$. Thus T is normal.

Now let T be a log-hyponormal operator. Since T is invertible there exists $c>0$ such that $c|T^*|\geqslant I$, so $\log(c|T^*|)\geqslant 0$. Due to $\log|T|\geqslant\log|T^*|=U(\log|T|)U^*$ we have $\log(c|T|)\geqslant\log(c|T^*|)=U\log(c|T|)U^*$ by Lemma 2.2 and equality (2). The rest of the proof is similar to the argument above and the proof of Theorem 2.1 so we omit it. □

In the sequel we are going to present a relationship between an operator and its Aluthge transform. We essentially apply the following lemma:

Lemma 2.4. *(See [12].) Let $T, S \in \mathbb{B}(\mathscr{H})$. Then the following assertions are equivalent:*

(i) *If $TX = XS$, then $T^*X = XS^*$ for any $X \in \mathbb{B}(\mathscr{H})$.*
(ii) *If $TX = XS$ where $X \in \mathbb{B}(\mathscr{H})$, then $\overline{R(X)}$ reduces T, $(\ker X)^\perp$ reduces S, and operators $T|_{\overline{R(X)}}$ and $S|_{(\ker X)^\perp}$ are normal.*

Theorem 2.5. *Let $T \in \mathbb{B}(\mathscr{H})$ be an invertible operator and $T = U|T|$ be the polar decomposition of T. Let $\mathrm{sp}(U)$ be contained in some open semicircle. Then $\widetilde{T}$ is normal if and only if so is T.*

Proof. Assume that $\widetilde{T}$ is normal. Hence $\widetilde{T}X = X\widetilde{T}$ implies $\widetilde{T}^*X = X\widetilde{T}^*$ for any $X \in \mathbb{B}(\mathscr{H})$ by Fuglede–Putnam commutativity theorem. We first show that $TX = XT$ implies $T^*X = XT^*$ for any $X \in \mathbb{B}(\mathscr{H})$. Let $X \in \mathbb{B}(\mathscr{H})$ and $TX = XT$. Then $U|T|X = XU|T|$, whence

$$\begin{aligned}\widetilde{T}\big(|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}\big) &= |T|^{\frac{1}{2}}\big(U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}X\big)|T|^{\frac{-1}{2}} \\ &= |T|^{\frac{1}{2}}\big(X|T|^{\frac{-1}{2}}|T|^{\frac{1}{2}}U|T|\big)|T|^{\frac{-1}{2}} \\ &= \big(|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}\big)\widetilde{T}. \end{aligned} \tag{5}$$

By (5) and the assumption with $|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}$ instead of X we have

$$\begin{aligned}|T|^{\frac{1}{2}}U^*|T|X|T|^{\frac{-1}{2}} &= |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}\big(|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}\big) = \widetilde{T}^*\big(|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}\big) \\ &= \big(|T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}\big)\widetilde{T}^* = |T|^{\frac{1}{2}}X|T|^{\frac{-1}{2}}|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}} \\ &= |T|^{\frac{1}{2}}XU^*|T|^{\frac{1}{2}}.\end{aligned}$$

So that

$$U^*|T|X = XU^*|T| \tag{6}$$

and $|T|X|T|^{-1} = UXU^*$. Therefore

$$|T|X|T|^{-1} = U^*\big(U|T|X\big)|T|^{-1} = U^*\big(XU|T|\big)|T|^{-1} = U^*XU.$$

Thus $UXU^* = U^*XU$, whence $U^2X = XU^2$.

Now we use the Beck and Putnam argument used in [3]. We replace U by $e^{\alpha}U$ if it is necessary and assume that $\mathrm{sp}(U)$ is contained in the set $\{e^{i\lambda}\colon\ \varepsilon < \lambda < \pi - \varepsilon\}$ for some $\varepsilon > 0$. Let $U = \int_{\varepsilon}^{\pi-\varepsilon} e^{i\lambda}\,\mathrm{d}E(\lambda)$ be the spectral decomposition of U. One has $U^2 = \int_{2\varepsilon}^{2\pi-2\varepsilon} e^{i\lambda}\,\mathrm{d}F(\lambda)$, where $F(\lambda) = E(\frac{\lambda}{2})$. By $U^2X = XU^2$ we have $U^{2n}X = XU^{2n}$ for every $n \in \mathbb{Z}$, so $U^{2n} = \int_{2\varepsilon}^{2\pi-2\varepsilon} e^{in\lambda}\,\mathrm{d}F(\lambda)$. Hence $f(U^2)X = Xf(U^2)$ for every f in the set of all bounded Borel-measurable complex-valued functions on $\{z\colon\ |z| = 1\}$ since $\{e^{int}\}$ is complete on the interval $0 \leqslant t \leqslant 2\pi$. Hence, by spectral resolution for normal operator U, $F(\lambda)X = XF(\lambda)$, whence $E(\lambda)X = XE(\lambda)$ and this implies again that $UX = XU$ and clearly this implies that

$$U^*X = XU^*. \tag{7}$$

From (6) and (7) we obtain

$$|T|X = U\big(U^*|T|X\big) = U\big(XU^*|T|\big) = U\big(U^*X\big)|T| = X|T|. \tag{8}$$

From (7) and (8) we deduce that $T^*X = |T|U^*X = X|T|U^* = XT^*$ as desired. We have shown that $TX = XT$ implies $T^*X = XT^*$ for any $X \in \mathbb{B}(\mathscr{H})$. It follows from Lemma 2.4(ii) for $X = I$ that T is normal.

The reverse is easy. In fact if T is normal, then $\widetilde{T} = T$. □

Acknowledgement

The authors would like to sincerely thank the referee for very useful comments and suggestions.

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integral Equations Operator Theory 13 (1990) 307–315.
[2] A. Aluthge, D. Wang, w-hyponormal operators, Integral Equations Operator Theory 36 (2000) 1–10.
[3] W.A. Beck, C.R. Putnam, A note on normal operators and their adjoints, J. London Math. Soc. 31 (1956) 213–216.
[4] P.B. Duggal, On nth roots of normal operators, Bull. London Math. Soc. 25 (1993) 74–80.
[5] M.R. Embry, nth roots of normal contractions, Proc. Amer. Math. Soc. 19 (1968) 63–68.
[6] I.H. Jeon, K. Tanahashi, A. Uchiyama, On quasisimilarity for log-hyponormal operator, Glasgow Math. J. 46 (2004) 169–176.
[7] T. Furuta, Invitation to Linear Operator, Taylor and Francis, London, New York, 2001.

[8] T. Okayasu, Y. Ueta, A condition under which $B = A = U^*BU$ follows from $B \geqslant A \geqslant U^*BU$, Proc. Amer. Math. Soc. 135 (2007) 1399–1403.
[9] C.R. Putnam, On square roots of normal operators, Proc. Amer. Math. Soc. 8 (1957) 768–769.
[10] H. Radjavi, P. Rosenthal, On roots of normal operators, J. Math. Anal. Appl. 34 (1971) 653–664.
[11] J.G. Stampfli, Hyponormal operators, Pacific J. Math. 12 (1962) 1453–1458.
[12] K. Takahashi, On the converse of Fuglede–Putnam theorem, Acta Sci. Math. (Szeged) 43 (1981) 123–125.
[13] K. Tanahashi, On log-hyponormal operators, Integral Equations Operator Theory 34 (1999) 364–372.
[14] A. Uchiyama, K. Tanahashi, Fuglede–Putnam's theorem for p-hyponormal or log-hyponormal operators, Glasgow Math. J. 44 (2002) 397–410.