C. R. Acad. Sci. Paris, Ser. I 349 (2011) 441–443

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential Geometry/Differential Topology

Quantitative Morse–Sard Theorem via Algebraic Lemma

Le théorème de Sard quantitatif via le lemme algébrique de Gromov

David Burguet

CMLA, ENS Cachan, 61, avenue du Président Wilson, 94230 Cachan, France

ARTICLE INFO

Article history:
Received 30 November 2010
Accepted after revision 18 January 2011

Presented by Jean-Pierre Demailly

ABSTRACT

We give a short proof of the so-called Quantitative Morse–Sard Theorem as an application of Gromov's Algebraic Lemma.

RÉSUMÉ

Nous donnons une preuve courte du théorème quantitatif de Morse–Sard comme application du lemme algébrique de Gromov.

1. Introduction

The classical Morse–Sard Theorem states that the set of critical values of a sufficiently smooth map $f:\mathbb{R}^n\to\mathbb{R}^m$ has zero Lebesgue measure. Using polynomial approximation and tools of semi-algebraic geometry Y. Yomdin [6] proved a quantitative version of this result which gives in particular an upper bound on the upper box dimension of the set of critical values.

In the whole Note the Euclidean spaces $\mathbb{R}^d$ with $d\geqslant 1$ are endowed with the usual Euclidean norm. Moreover the norm of multilinear maps on such Euclidean spaces will be the associated operator norm. When $f:]0,1[^n\to\mathbb{R}^m$ is a $\mathcal{C}^k$ map, we will consider the map $\Lambda^i f$ induced by f on the ith exterior power of $\mathbb{R}^n$ with $1\leqslant i\leqslant m$. The norm of $\Lambda^i f(x)$ is the growth under f of infinitesimal i-volume at x. In particular the n-volume $\mathrm{Vol}_n(f(]0,1[^n))$ of $f(]0,1[^n)$ is bounded from above by $\|\Lambda^n f\|_\infty$. The norm $\|\Lambda^i f(x)\|$ is also the product of the ith maximal eigenvalues of the square root of the differential map D_xf. We will denote by $\|f\|_k$ the supremum norm of the kth-derivative of f and by $\Delta(f,\nu)$ the critical values $f(x)$ of f such that the rank of the differential map D_xf is less than or equal to ν with $0\leqslant\nu<\min(m,n)$.

The ϵ-entropy $M(X,\epsilon)$ of a subset X of $\mathbb{R}^m$ is the minimal cardinality of collections of balls of radius ϵ covering X. The upper box dimension $\dim_b(X)$ of X is then just defined by

$$\dim_b(X)=\limsup_{\epsilon\to 0}-\frac{\log M(X,\epsilon)}{\epsilon}.$$

The main goal of this Note is to provide a short proof of the following Quantitative Morse–Sard Theorem due to Yomdin. We refer to [7] for further developments around Morse–Sard Theorem, in particular the question of optimality of this result.

E-mail address: David.Burguet@cmla.ens-cachan.fr.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.01.019

Theorem 1.1. *Let $f:]0,1[^n \to \mathbb{R}^m$ be a $\mathcal{C}^k$ map with $k \in \mathbb{N} \setminus \{0\}$, then for all $0 \leqslant \nu < \min(n,m)$*

$$M\big(\epsilon, \Delta(f,\nu)\big) \leqslant C \sum_{j=0}^{\nu} \epsilon^{-j-\frac{n-j}{k}} \frac{\|\Lambda^j f\|_\infty}{\|f\|_k^{\frac{n-j}{k}}}$$

where C is bounded by a function depending only on n, m, k.[1] *In particular* $\dim_b(\Delta(f,\nu)) \leqslant \nu + \frac{n-\nu}{k}$.

2. Semi-algebraic tools

A subset A of $\mathbb{R}^d$ is said to be semi-algebraic if it can be written as a finite union of polynomial equalities and inequalities. Such a presentation is not necessarily unique. A map $f: A \subset \mathbb{R}^d \to \mathbb{R}^e$ is semi-algebraic if its graph is semi-algebraic.

To estimate the algebraic complexity of a semi-algebraic set we define its degree as the minimum over all its presentations of the sum of the degree of the polynomials (counted with multiplicity) involving in the presentation. The degree of a semi-algebraic map is the degree of its graph. We will use the following smooth decomposition of semi-algebraic maps:

Lemma 2.1. *(See [5, Thm. 3.2, p. 115].) For any semi-algebraic map $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ there exists a partition of A in semi-algebraic manifolds $(A_j)_{j=1,\dots,N}$ such that $f|_{A_j}$ is $\mathcal{C}^1$ for all $j = 1,\dots,N$. Moreover N and the degree of A_j are bounded by a function depending only on n and the degree of A.*

A key tool of our proof is the following $\mathcal{C}^1$ reparametrization theorem also known as Gromov's Algebraic Lemma. The Algebraic Lemma bounds the differential complexity of a semi-algebraic set by its algebraic complexity:

Lemma 2.2. *(See [2, Lemma 3.3, p. 232].) Let $A \subset [0,1]^n$ be a semi-algebraic set of dimension l, then there exist semi-algebraic $\mathcal{C}^1$ embeddings $(\phi_i :]0,1[^{l_i} \to A)_{i=1,\dots,N}$ with $0 \leqslant l_i \leqslant l$ (by convention $]0,1[^0$ is the singleton $\{0\}$) such that $\|\phi_i\|_1 \leqslant 1$ and $\bigcup_i \phi_i(]0,1[^{l_i}) = A$. Moreover N and the degree of the reparametrizations ϕ_i are bounded by a function depending only on n and the degree of A.*

We observe that the parametrizations ϕ_i are uniformly continuous on open unit squares so that they can be extended continuously on the closure of these squares.

When $n = 1$ the lemma is trivial as the connected components of A are just intervals of length less than one which can be reparametrized by affine contractions (the number of connected components is in this case obviously bounded from above by the degree of A). Complete proofs of the above lemma can be found in [1] and [4]. The lemma holds also true if we replace the norm $\|\,\|_1$ by $\max_{l=1,\dots,k} \|\,\|_l$ for any k.

To compare the ϵ-entropy and the volume of semi-algebraic sets we will need the following Semi-algebraic Choice. This result will be only used to prove the Quantitative Morse–Sard Theorem for smooth functions $f: \mathbb{R}^n \to \mathbb{R}^m$ with n larger than m.

Lemma 2.3. *(See [5, Prop. 1.2, p. 94].) Let $f: A \subset [0,1]^n \to \mathbb{R}^m$ be a semi-algebraic map, then there exists a semi-algebraic set $B \subset A$ of dimension less than m with $f(B) = f(A)$. Moreover the degree of B is bounded by a function depending only on n, m and the degree of f.*

3. ϵ-entropy and volume of semi-algebraic sets

We first relate the ϵ-entropy of images of smooth semi-algebraic maps with their volume. Refinements of such results can be found in [3].

Lemma 3.1. *Let $f:]0,1[^n \to \mathbb{R}^m$ be a semi-algebraic map which extends continuously on $[0,1]^n$ then*

$$M\big(\epsilon, f\big([0,1]^n\big)\big) \leqslant C \sum_{0 \leqslant i \leqslant m} \sum_{A_i} \mathrm{Vol}_i\big(f(A_i)\big)\epsilon^{-i}$$

where the second sum holds over semi-algebraic sets $A_i \subset [0,1]^n$ of dimension less than m with $\deg(A_i) \leqslant C$ and $\sharp\{A_i\} \leqslant C$. Moreover C is bounded by a function depending only on n, m and the degree of f.

Proof. We argue by induction on n. According to the Algebraic Lemma and Lemma 2.1 one can assume f to be $\mathcal{C}^1$. By applying the Semi-algebraic Choice we only need to consider the case $n \leqslant m$. Let $r(x)$ be the rank of $D_x f$. Since the set

[1] We do not intend to get estimates of the "universal" constant C (in the following proofs C will always designate some constant depending only on n, m, k when we do not precise its meaning).

$\{x \in]0,1[^n,\ r(x) = k\}$ with $0 \leqslant k \leqslant n$ is semi-algebraic one can assume by applying again the Algebraic Lemma that r is constant on $]0,1[^n$. Similarly it is enough to consider the case where $\|\pi \circ D_x f(u)\| \geqslant \sqrt{\frac{r}{m}}\|D_x f(u)\|$ $(*)$ for all $x \in]0,1[^n$ and for all $u \in \mathbb{R}^n$ with π the projection on the r first coordinates of $\mathbb{R}^m$. In particular $\pi \circ f$ is an open map.

Let O be the union of balls of $\mathbb{R}^m$ of radius 3ϵ covering $f(\partial[0,1]^n)$ with cardinality $M(3\epsilon, f(\partial[0,1]^n))$. Then we consider a collection $\mathcal{U}$, with minimal cardinality, of balls of radius ϵ whose union covers $f([0,1]^n) \setminus O$. Observe that these balls do not intersect $f(\partial[0,1]^n)$. By Vitali's Covering Lemma, one can extract from $\mathcal{U}$ a finite collection $\mathcal{V}$ of disjoints balls such that the balls of radius 3ϵ at the same centers are covering $f([0,1]^n) \setminus O$. In particular we get the following upper bound $M(3\epsilon, f([0,1]^n)) \leqslant M(3\epsilon, f(\partial[0,1]^n)) + \sharp\mathcal{V}$. Moreover by assumption $(*)$ the projection of the intersection of any ball B in $\mathcal{U}$ with $f(]0,1[^n)$ contains a ball of $\mathbb{R}^r$ of radius larger than $C^{-1}\epsilon$. Therefore $\mathrm{Vol}_r(B \cap f([0,1]^n)) \geqslant \mathrm{Vol}_r(\pi(B \cap f([0,1]^n))) \geqslant C\epsilon^r$. Finally we conclude that $\mathrm{Vol}_r f(]0,1[^n) \geqslant C\epsilon^r \sharp\mathcal{V}$ and then $M(3\epsilon, f([0,1]^n)) \leqslant M(3\epsilon, f(\partial[0,1]^n)) + C\epsilon^{-r}\mathrm{Vol}_r f(]0,1[^n)$. This concludes the proof by induction as the map f can be smoothly reparametrized on the boundary of $[0,1]^n$ from $]0,1[^{n-1}$ according to the Algebraic Lemma and Lemma 2.1. □

4. Proof of the Quantitative Morse–Sard Theorem

We divide the unit cube into subcubes of size $\alpha := (\frac{\epsilon}{\|f\|_k})^{\frac{1}{k}}$. We consider such a subcube $\mathcal{S}$ and we denote by $y_{\mathcal{S}}$ its left-bottom corner. We set $g_{\mathcal{S}} = f(\alpha. + y_{\mathcal{S}})$. Let $P_{\mathcal{S}}$ be the Taylor polynomial of order r of $g_{\mathcal{S}}$ at $(\frac{1}{2},\ldots,\frac{1}{2})$. By Taylor formula we have

$$\|g_{\mathcal{S}} - P_{\mathcal{S}}\|_\infty \leqslant \|g_{\mathcal{S}}\|_k \leqslant \alpha^k\|f\|_k \leqslant \epsilon \quad \text{and similarly} \quad \|D_. g_{\mathcal{S}} - D_. P_{\mathcal{S}}\|_\infty \leqslant \epsilon \tag{1}$$

By Lemma 6.2 of [7] we have for all $x \in]0,1[^n$ and for all $i = 1,\ldots,m$ (with the convention $\|\Lambda^0 g_{\mathcal{S}}\| = 1$)

$$\left\|\Lambda^i P_{\mathcal{S}}(x)\right\| \leqslant C\sum_{j=0}^{i}\left\|\Lambda^j g_{\mathcal{S}}(x)\right\|\|D_x g_{\mathcal{S}} - D_x P_{\mathcal{S}}\|^{i-j} \tag{2}$$

By combining the two last inequalities (1) and (2) we get $\|\Lambda^i P_{\mathcal{S}}(x)\| \leqslant C\sum_{j=0}^{i}\|\Lambda^j g_{\mathcal{S}}(x)\|\epsilon^{i-j}$ for all $x \in]0,1[^n$ and for all $i = 1,\ldots,m$. Let A be the intersection $\bigcap_{i=1,\ldots,\min(m,n)}\{\|\Lambda^i P_{\mathcal{S}}\|_\infty \leqslant C\sum_{j=0}^{\min(\nu,i)}\|\Lambda^j g_{\mathcal{S}}\|_\infty\epsilon^{i-j}\}$ so that $\Delta(f|_{\mathcal{S}},\nu)$ is a subset of the ϵ-neighborhood of $P_{\mathcal{S}}(A)$. We apply the Algebraic Lemma to the semi-algebraic set A. Let $(\phi_j :]0,1[^{l_j} \to A)_{j=1,\ldots,N}$ be the $\mathcal{C}^1$ semi-algebraic embeddings with $\|\phi_j\|_1 \leqslant 1$ reparametrizing A. By Lemma 3.1 we get

$$M\big(2\epsilon, \Delta(f|_{\mathcal{S}},\nu)\big) \leqslant M\big(\epsilon, P_{\mathcal{S}}(A)\big) \leqslant \sum_{j=1}^{N} M\big(\epsilon, P_{\mathcal{S}}\circ\phi_j\big(]0,1[^{l_j}\big)\big) \leqslant C\sum_{i,j}\left\|\Lambda^i(P_{\mathcal{S}}\circ\phi_j)\right\|_\infty\epsilon^{-i}$$

As $\|\phi_j\|_1 \leqslant 1$ we have $\|\Lambda^i(P_{\mathcal{S}}\circ\phi_j)\|_\infty \leqslant \|\Lambda^i P_{\mathcal{S}}|_A\|_\infty$ and it follows from the definition of A that for $i = 1,\ldots,\min(m,n)$:

$$\left\|\Lambda^i P_{\mathcal{S}}|_A\right\|_\infty \leqslant C\sum_{j=0}^{\min(\nu,i)}\left\|\Lambda^j g_{\mathcal{S}}\right\|_\infty\epsilon^{i-j}$$

Recall now that $g_{\mathcal{S}} = f(\alpha. + y_{\mathcal{S}})$. Therefore $\Lambda^j g_{\mathcal{S}} = \alpha^j\Lambda^j f(\alpha. + y_{\mathcal{S}})$ for all $j = 1,\ldots,\min(m,n)$. We conclude

$$M\big(2\epsilon, \Delta(f,\nu)\big) \leqslant CN\alpha^{-n}\sum_{j=0}^{\nu}\alpha^j\epsilon^{-j}\left\|\Lambda^j f\right\|_\infty \leqslant C\sum_{j=0}^{\nu}\epsilon^{-j-\frac{n-j}{k}}\frac{\|\Lambda^j f\|_\infty}{\|f\|_k^{\frac{n-j}{k}}}$$

Remark. When $m = 1$ the proof is very easy. Indeed the maps $P_{\mathcal{S}}\circ\phi_j$ have derivative less than ϵ and their image is then an interval of length less than ϵ. Therefore in this case $M(2\epsilon, \Delta(f,\nu))$ is directly bounded from above by the number of such maps. In particular we do not need to use Lemma 3.1.

References

[1] D. Burguet, A proof of Yomdin–Gromov's algebraic lemma, Israel J. Math. 168 (2008) 291–316.
[2] M. Gromov, Entropy, homology and semi-algebraic geometry, in: Séminaire Bourbaki, vol. 1985/1986, Astérisque 145–146 (1987) 225–240.
[3] L.D. Ivanov, in: A.G. Vituskin (Ed.), Variatsii Mnozhestv i Funktsii (Variations of Sets and Functions), Izdat. Nauka, Moscow, 1975 (in Russian).
[4] J. Pila, A.J. Wilkie, The rational points of a definable set, Duke Mathematical Journal 133 (2006) 591–616.
[5] Lou van den Dries, Tame Topology and O-minimal Structures, Cambridge University Press, 1998.
[6] Y. Yomdin, The geometry of critical and near-critical values of differentiable mappings, Math. Ann. 264 (1983) 495–515.
[7] Y. Yomdin, G. Comte, Tame geometry with application in smooth analysis, in: Lecture Notes in Mathematics, vol. 1834, Springer-Verlag, Berlin, 2004.