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Hyperbolic embeddability of locally complete almost complex submanifolds

Plongement hyperbolique des sous variétés presques complexe localement complètes

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ABSTRACT

In this Note, we generalize to the almost complex setting, a theorem of Zaidenberg (1983) [13] and Thai (1991) [12] by giving a characterization on hyperbolic embeddability of a locally complete and relatively compact almost complex submanifold in terms of extension of pseudo-holomorphic disks from the punctured unit disk and of limit J-complex lines.

RÉSUMÉ

Dans cette Note, on généralise dans le cas presque complexe un théorème de Zaidenberg (1983) [13] et Thai (1991) [12] en donnant une caractérisation des variétés presque complexe relativement compacte, hyperboliquement plongés et localement complètes en terme d'extension des courbes pseudo-holomorphes et des limites de droites J-complexes.

Version française abrégée

Une variété presque complexe (N, J) est une variété lisse réelle équipée d'une structure presque complexe J. En utilisant les courbes pseudo-holomorphes, on peut définir la pseudo-distance de Kobayashi d_N^J et la pseudo-métrique de Kobayashi–Royden K_N^J.

On dit que (N, J) est *hyperbolique* si d_N^J est une distance. Une variété presque complexe hyperbolique (N, J) est dite *hyperbolique complète* si (N, d_N^J) est un espace complet.

Soit (M, J) une sous-variété presque complexe d'une variété presque complexe (N, J). On dit que (M, J) est *localement hyperbolique complète* si tout point p dans $\overline{M}$ admet un voisinage V dans N tel que $(M \cap V, J)$ est hyperbolique complet.

Rappelons que (M, J) est dite *hyperboliquement plongée* dans (N, J) si pour tous $x, y \in \overline{M}$, $x \neq y$, il existe des voisinages ouverts U de x et V de y telles que $d_M^J(M \cap U, M \cap V) > 0$.

On dit que (M, J) possède la propriété de *Δ^*-prolongement pour* (N, J) si toute courbe pseudo-holomorphe de Δ^* dans (M, J) se prolonge en une courbe pseudo-holomorphe de Δ dans (N, J), où Δ désigne le disque unité et $\Delta^* = \Delta \setminus \{0\}$.

Joo [5] a prouvé, en utilisant le lemme de monotonie de Gromov [9], que (M, J) possède la propriété de Δ^*-prolongement pour (N, J) si (M, J) est relativement compact et hyperboliquement plongée dans (N, J). Dans [4], on a

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donné une preuve élémentaire de ce résultat en montrant que la courbe pseudo-holomorphe considérée définie sur le disque épointé a une énergie finie, donc se prolonge.

D'abord on donne une autre preuve du théorème de prolongement en utilisant les fonctions de Chirka, ensuite on démontre le résultat principal suivant de cette Note :

Théoremè 1. *Soit (M, J) une sous variété localement hyperbolique complète et relativement compacte d'une variété presque complexe (N, J). Alors les propositions suivantes sont équivalentes :*

(i) *(M, J) est hyperboliquement plongé dans (N, J).*
(ii) *(M, J) possède la propriété de Δ^*-prolongement pour (N, J) et ∂M ne contient aucune limite de droites J-complexes.*
(iii) *M ne contient aucune droite J-complexe et ∂M ne contient aucune limite de droites J-complexes.*

De plus, si l'une des ces conditions est satisfaite, alors (M, J) est hyperbolique complète.

1. Introduction and definitions

For every $r > 0$, we set $\Delta_r = \{z \in \mathbb{C},\ |z| < r\}$, $\Delta = \Delta_1$ is the unit disk and $\Delta^* = \Delta \setminus \{0\}$.

An almost complex manifold (N, J) is a smooth real manifold equipped with an almost complex structure J, that is a $\mathcal{C}^\infty$-field of complex linear structures on the tangent bundle TN. Using pseudo-holomorphic curves, we can define the Kobayashi pseudo-distance d_N^J on (N, J) and the Kobayashi–Royden infinitesimal pseudo-metric K_N^J on TN.

We say that (N, J) is *hyperbolic* if d_N^J is a distance. A hyperbolic almost complex manifold (N, J) is *complete hyperbolic* if it is Cauchy complete with respect to d_N^J.

Let (M, J) be an almost complex submanifold of an almost complex manifold (N, J).

We say that (M, J) is *locally complete hyperbolic* if every $p \in \overline{M}$ has a neighborhood V in N such that $(M \cap V, J)$ is complete hyperbolic. For instance, it is known that given (M, J) an almost complex manifold of real dimension 4 and D an immersed pseudo-holomorphic curve in M, then $(M \setminus D, J)$ is locally complete hyperbolic, see [1].

Recall that (M, J) is *hyperbolically embedded* in (N, J) if for $x, y \in \overline{M}$, $x \neq y$, there exist open neighborhoods U of x and V of y such that $d_M^J(M \cap U, M \cap V) > 0$. Some criteria for hyperbolic embedding are proved in [4]. Namely, we prove that if M is relatively compact in N, then (M, J) is hyperbolically embedded in (N, J), if and only if for every length function G on N, there exists a positive constant c such that $K_M^J \geqslant c.G$.

We say that (M, J) has the Δ^*-*extension property* for (N, J) if every pseudo-holomorphic curve from Δ^* into (M, J) can be extended to a pseudo-holomorphic curve from Δ into (N, J).

Joo [5] proved that (M, J) has the Δ^*-*extension property* for (N, J) if (M, J) is relatively compact and hyperbolically embedded in (N, J). His proof is based on the so-called Gromov's monotonicity lemma [9]. In [4], the authors gave an elementary proof of that result by showing that the considered pseudo-holomorphic curve defined on the punctured disk has a finite energy and then extends. We provide here another proof based on "Chirka's functions" with prescribed poles. We should mention that this gives an alternative approach to the standard case.

Let $f : \Delta^* \to (M, J)$ be a pseudo-holomorphic curve. We are interested when this map extends to a pseudo-holomorphic curve on all of Δ. In other words, we seek conditions when f has removable singularity at 0.

Following Lang (see [8], p. 40), we consider the following conditions:

J-KW 1. (M, J) is hyperbolically embedded in (N, J), and there exists a sequence $\{z_n\}$ in Δ^* such that $z_n \to 0$ and $\{f(z_n)\}$ converges to a point $p \in \overline{M}$.

We note that the condition about the existence of the sequence $\{z_n\}$ is automatically satisfied if M is relatively compact in N.

J-KW 2. The map f has a removable singularity at 0, that is, f extends to a pseudo-holomorphic curve in a neighborhood of 0.

Theorem 1. *We have the implication*

$$\textbf{J-KW 1} \quad \Longrightarrow \quad \textbf{J-KW 2}.$$

In particular, if (M, J) is a relatively compact almost complex submanifold hyperbolically embedded in an almost complex manifold (N, J), then every pseudo-holomorphic curve $f : \Delta^* \to M$ extends to a pseudo-holomorphic curve $\tilde{f} : \Delta \to N$, i.e., (M, J) has the Δ^*-extension property for (N, J).

We mention that such a theorem, in the complex case, is first proved by Kwack [7]. Her proof is based on winding numbers arguments due to Grauert–Reckzigel. Later Noguchi [10] proved this theorem, his arguments exploited the lower bound formula for the area of analytic varieties in terms of the Lelong numbers.

2. Proofs required, and main result

Our proof is based on two ingredients. The first is due to Chirka, see [2].

Lemma 2. *Given (M, J) an almost complex manifold, then for every $p \in M$ there exist relatively compact local coordinate neighborhoods U, W of p such that $W \Subset U$ and positive constants A and B such that for every $q \in \overline{W}$, the mapping $z \mapsto \log|z-q| + A|z-q| + B|z|^2$ is J-plurisubharmonic on U.*

Consequently, by adding such functions, we can construct a J-plurisubharmonic function on U with finitely many poles in $\overline{W}$.

The second is the following well-known lemma in the complex case and still valid in the almost complex case.

Lemma 3. *Let (M, J) be an almost complex submanifold hyperbolically embedded in an almost complex manifold (N, J), $f : \Delta^* \to (M, J)$ be a pseudo-holomorphic curve and let $p \in \overline{M}$. If (z_n) is a sequence in Δ^* such that $f(z_n) \to p$, then $f(\sigma_n) \to p$, where $\sigma_n = \{z \in \Delta\colon\ |z| = |z_n|\}$.*

Proof of Theorem 1. Assume **J-KW 1**. Let U and W be relatively compact local coordinate neighborhoods of p such that $\overline{W} \subset U$, as is Lemma 2.

It will suffice to prove that there exists $r > 0$ such that $f(\Delta_r^*) \subset W$, thus showing that f extends to a continuous map from Δ to N and by a result of Sikorav ([11], p. 169) it is known that if f is continuous everywhere, differentiable and pseudo-holomorphic except on a discrete subset, then it is differentiable and thus pseudo-holomorphic.

Assume there is no such r. Using Lemma 3, we have $f(\sigma_n) \to p$, where $\sigma_n = \{z \in \Delta\colon\ |z| = |z_n|\}$. This implies that $f(\sigma_n) \subset W$, for sufficiently large n. Let a_n, b_n be positive numbers with $a_n < |z_n| < b_n$ such that the annulus defined by $a_n < |z| < b_n$ is the largest whose image under f is contained in W. We let

$$\alpha_n(t) = a_n \exp 2i\pi t \quad \text{and} \quad \beta_n(t) = b_n \exp 2i\pi t, \quad 0 \leqslant t \leqslant 1$$

be the two circles bounding the open annulus. Then

$$f(\alpha_n) \quad \text{and} \quad f(\beta_n) \subset \overline{W},$$

but these images of the two circles α_n and β_n are not contained in W. Since $\overline{W}$ is compact, using Lemma 3, we may assume after sub-sequencing that $f(\alpha_n)$ converges to $q \in \partial W$ and we have $q \neq p$.

Let $\mathcal{A}_n = \{z \in \mathbb{C}\colon\ a_n \leqslant |z| \leqslant |z_n|\}$ and φ be a J-plurisubharmonic on U satisfying $\varphi^{-1}(-\infty) = \{p, q\}$. Such function exits by Lemma 2. Hence, for $K > 0$ and for n sufficiently large, we have

$$f(\partial\mathcal{A}_n) \subset \bigl\{z \in U\colon\ \varphi(z) \leqslant -K\bigr\}.$$

Applying the standard maximum principle, we deduce that $f(\mathcal{A}_n) \subset \{z \in U\colon\ \varphi(z) \leqslant -K\}$. Finally, we get a contradiction. Since p and q are not in the same connected component for K sufficiently large. □

Remark 1. We can also consider the annulus $\mathcal{B}_n = \{z \in \mathbb{C}\colon\ |z_n| \leqslant |z| \leqslant b_n\}$ and we remark, after sub-sequencing, that the sequence $f(\beta_n)$ converges to $q' \in \partial W$ and we have $q' \neq p$.

Next, we generalize a theorem of Zaidenberg (see [13], Theorem 2.1) to the almost complex case.

Theorem 4. *Let (M, J) be a locally complete hyperbolic and relatively compact submanifold of an almost complex manifold (N, J). If M contains no J-complex lines and ∂M contains no limit J-complex lines, then (M, J) is hyperbolically embedded in (N, J).*

Here by a *limit J-complex line* of ∂M we mean a non-constant entire J-holomorphic curve with values in ∂M which can be approximated on each Δ_R by J-holomorphic curves from Δ_R into M.

The proof of Theorem 4 is an easy adaptation of the ideas of Green [3] and Zaidenberg [13] and contains a number of intermediate steps, which we sketch. Using a special covering, we introduce a differential metric on M called the Kobayashi–Royden–Green (KRG) metric. We should mention, that using the KRG-metric, the author and Haggui proved the stability of hyperbolic embeddedness under small perturbation of the almost complex structure, see Theorem 3.3 in [4].

Proof. Let G be a fixed length function on N. Cover $\overline{M}$ by open subsets $U_1, \dots, U_\ell$, such that for each $\nu = 1, \dots \ell$, we have (U_ν, J) and $(U_\nu \cap M, J)$ are complete hyperbolic. For $\xi_x \in TM_x$, we put $G_0(\xi_x) = \min\{K_{U_\nu \cap M}(\xi_x)\}$ where the minimum is taken over those ν so that $x \in U_\nu$. One can easily prove that G_0 is continuous and complete. Now, choose $\varepsilon > 0$ such that for each $x \in N$, $B_G(x, \varepsilon)$ the G-ball of radius ε in N lies inside one of (U_ν). Define $G_\varepsilon = \max(G, \frac{\varepsilon}{3}G_0)$. Clearly, G_ε is a complete metric on M. Now, we prove the following assertion: there exists a positive constant c such that $K_M^J \geqslant cG_\varepsilon$. Otherwise, there is a monotone increasing sequence (r_n) of positive numbers tending to $+\infty$ and a sequence of pseudo-holomorphic curves $f_n : \Delta_{r_n} \to (M, J)$ such that $f_n'(0) = \xi_n$ and $|\xi_n|_{G_\varepsilon} = 1$. Applying Brody's reparametrization theorem, we obtain a sequence of pseudo-holomorphic curves $g_n : \Delta_{r_n} \to (M, J)$ such that $|g_n'(z)|_{G_\varepsilon} \leqslant \frac{r_n^2}{r_n^2 - |z|^2}$ on Δ_{r_n} and $|g_n'(0)|_{G_\varepsilon} = 1$. Hence, we can extract a subsequence, denoted also by (g_n) which converges uniformly with all derivatives on compact sets to a pseudo-holomorphic curve $g : \mathbb{C} \to (N, J)$. As in the complex case (see [13], Lemma 2.7), we prove that $|g'(0)|_G = 1$ and $|g'(z)|_G \leqslant 4/3$. Consequently, g is a J-complex line. Now, we prove that either $g(\mathbb{C}) \subset \partial M$ or $g(\mathbb{C}) \subset M$ which contradicts the fact that M does not contain J-complex lines and ∂M does not contain limit J-complex lines. Suppose that there exists $z_0 \in \mathbb{C}$ such that $g(z_0) \in M$. Let z_1 be an arbitrary point of $\mathbb{C} \setminus \{z_0\}$. Since $|g_n'(z)|_{G_\epsilon} \leqslant 4/3$ for $z \in \Delta_{r_n/2}$, we have: $d_{G_\varepsilon}(g_n(z_1), g(z_o)) \leqslant d_{G_\varepsilon}(g_n(z_1), g_n(z_o)) + d_{G_\varepsilon}(g_n(z_0), g(z_o)) \leqslant 4/3|z_1 - z_0| + 2/3|z_1 - z_0|$. Hence, $g_n(z_1) \in B_{G_\epsilon}(g(z_0), 2|z_1 - z_0|)$. Since the metric G_ε is complete, so $B_{G_\epsilon}(g(z_0), 2|z_1 - z_0|) \Subset M$. Consequently, $g(z_1) = \lim g_n(z_1) \in M$ and finally $g(\mathbb{C}) \subset M$. □

We now prove the main result of this Note.

Theorem 5. *Let (M, J) be a locally complete hyperbolic and relatively compact submanifold of an almost complex manifold (N, J). Then the following are equivalent*:

(i) *(M, J) is hyperbolically embedded in (N, J).*
(ii) *(M, J) has the Δ^*-extension property for (N, J) and ∂M contains no limit J-complex lines.*
(iii) *M contains no J-complex lines and ∂M contains no limit J-complex lines.*

Moreover, if one of the above conditions holds, then (M, J) is complete hyperbolic.

Proof. (i) $\Rightarrow$ (ii). By Theorem 1, (M, J) has the Δ^*-extension property for (N, J). Thus it remains to show that ∂M contains no limit J-complex lines. We assume that there exists h a limiting J-complex line of ∂M. Let p, q be any pair of points in $h(\mathbb{C})$, then $p = h(a_1)$ and $q = h(a_2)$ where $a_1, a_2 \in \Delta_R$ for some positive radius R and there exists a sequence of J-holomorphic curves $f_n : \Delta_R \to M$ which converges uniformly to $h|\Delta_R$. By [4] (see Theorem 2.2), there exists a length function G on N such that $K_M^J \geqslant G$. Consequently, we have $d_G(f_n(a_1), f_n(a_2)) \leqslant d_M^J(f_n(a_1), f_n(a_2)) \leqslant d_{\Delta_R}(a_1, a_2)$. Let n tend to ∞, we get $d_G(p, q) \leqslant d_{\Delta_R}(a_1, a_2)$, let R tend to ∞, it follows that $p = q$ and h is a constant.

(ii) $\Rightarrow$ (iii). Assume that there exists a non-constant J-holomorphic curve $f : \mathbb{C} \to (M, J)$, say $f(1) \neq f(-1)$. Consider a holomorphic map g from Δ^* into $\mathbb{C}$ such that $g(1/n) = (-1)^n$. Clearly, the pseudo-holomorphic curve $f \circ g$ does not extend.

(iii) $\Rightarrow$ (i). Follows from Theorem 4.

Finally, we see that (M, J) is complete hyperbolic since it is locally complete hyperbolic and hyperbolically embedded in (N, J), see Kobayashi [6], Theorem 3.3.4, p. 72, which is still valid for almost complex manifolds. □

As an immediate consequence, we obtain the following characterization of the hyperbolicity of an almost complex compact manifold.

Corollary 6. *Let (M, J) be a compact almost complex manifold. Then the following are equivalent*:

(i) *(M, J) is hyperbolic.*
(ii) *M has the Δ^*-extension property.*
(iii) *M contains no J-complex lines.*

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