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Geometry

On a class of Riemannian metrics arising from Finsler structures

Sur une classe de métriques riemanniennes issues de structures finsleriennes

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A B S T R A C T

On the slit tangent bundle of Finsler manifolds, we introduce a class of metrics and study the relation between Levi-Civita connection, Vaisman connection, vertical foliation, and Reinhart spaces. We show that the Levi-Civita and the Vaisman connections induce the same connections in the structural bundle if and only if the base manifold is Landsbergian. Moreover every foliated Reinhart manifold reduces to a Riemannian manifold.

R É S U M É

Sur le fibré tangent d'une variéte finslerienne, nous introduisons une certaines classe de métriques et étudions la relation entre la connexion de Levi-Civita, la connexion de Vaisman, et les espaces de Reinhart. Nous montrons que les connexions de Levi-Civita et de Vaisman induisent les mêmes connexions dans le fibré structurel si seulement si la variété de base est de Landsberg. En outre, toute varété de Reinhart feuilletée se réduit à une variété riemannienne.

A Riemannian metric g on a smooth manifold M gives rise to several Riemannian metrics on the tangent bundle TM of M and maybe the best known example is the Sasaki metric. Although the Sasaki metric is defined in a natural way, it is very rigid. For example, the tangent bundle TM with the Sasaki metric is never locally symmetric unless the metric g on the base manifold M is flat [1,2]. For this reason, some mathematicians think on constructing a metric which save some more geometrical properties [5,8].

By using the Finsler metric on a manifold, we introduce a class of metrics on the slit tangent bundle of Finsler manifolds. Then we find the relation between Levi-Civita connection, Vaisman connection, vertical foliation, and Reinhart spaces. We show that the Levi-Civita and the Vaisman connections induce the same connections in the structural bundle if and only if the base manifold is Landsbergian. Then we prove that every foliated Reinhart manifold reduces to a Riemannian manifold.

Let (M, F) be a Finsler manifold. Consider $TM_0 = TM \setminus \{0\}$ and denote by $VTM_0 = \ker \pi_*$ the vertical vector bundle over TM_0, where π_* is the tangent mapping of the canonical projection $\pi : TM_0 \to M$ [3]. The Finsler metric $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is a Riemannian metric on VTM_0. The canonical nonlinear connection $HTM_0 = (N_i^j(x, y))$ of (M, F) is given by $N_i^j = \frac{\partial G^j}{\partial y^i}$, where $G^j = \frac{g^{jh}}{4}(\frac{\partial^2 F^2}{\partial y^h \partial x^k} y^k - \frac{\partial F^2}{\partial x^h})$. Then on any coordinate neighborhood $\mathfrak{u} \subset TM_0$ the vector fields $\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}\}_{i=1}^n$ form a basis for $\Gamma(HTM_{0|\mathfrak{u}})$. We have the following Lie brackets

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$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^k{}_{ij}\frac{\partial}{\partial y^k}, \qquad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = G^k_{ij}\frac{\partial}{\partial y^k}, \tag{1}$$

where $R^k{}_{ij} = \frac{\delta N^k_i}{\delta x^j} - \frac{\delta N^k_j}{\delta x^i}$ and $G^k_{ij} = \frac{\partial N^k_i}{\partial y^j}$.

Now, consider the energy density $2t(x, y) = F^2 = g_{ij}(x, y)y^i y^j$ defined by F and also the smooth functions $u, v : [0, \infty) \to \mathbb{R}$ such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for every t. The above conditions assure that the symmetric $(0, 2)$-type tensor field of TM_0, $G_{ij} = u(t)g_{ij} + v(t)y_i y_j$ is positive definite. The inverse of this matrix has the entries $H^{kl} = \frac{1}{u}g^{kl} + \omega(t)y^k y^l$, where (g^{kl}) are the components of the inverse of the matrix (g_{ij}) where $\omega(t) = -\frac{v}{u(u+2tv)}$. The components H^{kl} define symmetric $(0, 2)$-type tensor field of TM_0. It is easy to see that, if the matrix (G_{ij}) is positive definite then the matrix (H^{kl}) is positive definite too. We use the components H_{ij} of symmetric $(0, 2)$-type tensor field of TM_0 obtained from the components H^{kl} by "lowering" the indices $H_{ij} = g_{ik}H^{kl}g_{lj} = \frac{1}{u}g_{ij} + \omega y_i y_j$, where $y_i = g_{ik}y^k$. The following Riemannian metric may be considered on TM_0:

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G_{ij}, \qquad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = H_{ij}, \qquad G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0. \tag{2}$$

If $u = 1$ and $v = 0$, then (2) reduces to the Sasaki metric [5,6,4].

Lemma 1. *The Levi-Civita connection of the Riemannian metric G defined by* (2) *is as follows*

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = \frac{1}{u^2}\left(G^s_{ij} - F^s_{ij}\right)\frac{\delta}{\delta x^s} + \left(C^s_{ij} + \alpha_1 g_{ij}y^s - \alpha_2\left(y_i\delta^s_j + y_j\delta^s_i\right) + \alpha_3 y_i y_j y^s\right)\frac{\partial}{\partial y^s}, \tag{3}$$

$$\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = F^s_{ij}\frac{\delta}{\delta x^s} + \left(-u^2 C^s_{ij} + \alpha_4\left(y_j\delta^s_i + y_i\delta^s_j\right) + \alpha_5 y_i y_j y^s + \alpha_6 g_{ij}y^s + \frac{1}{2}R^s_{ij}\right)\frac{\partial}{\partial y^s}, \tag{4}$$

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = \left(C^s_{ij} + \alpha_7 y_i y_j y^s + \alpha_8 g_{ij}y^s + \alpha_2 y_i\delta^s_j + \alpha_9 y_j\delta^s_i + \frac{1}{2u}R_{ikj}H^{ks}\right)\frac{\delta}{\delta x^s} + \left(F^s_{ij} - G^s_{ij}\right)\frac{\partial}{\partial y^s}, \tag{5}$$

$$\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \left(C^s_{ij} + \alpha_7 y_i y_j y^s + \alpha_8 g_{ij}y^s + \alpha_2 y_j\delta^s_i + \alpha_9 y_i\delta^s_j + \frac{1}{2u}R_{jki}H^{ks}\right)\frac{\delta}{\delta x^s} + F^s_{ij}\frac{\partial}{\partial y^s}, \tag{6}$$

where $\alpha_1 = \frac{u'u+2tu'v+2wu^2(u+2tv)}{2u^2}$, $\alpha_2 = \frac{u'}{2u}$, $\alpha_3 = \frac{-2u'v+w'u^2(u+2tv)}{2u^2}$, $\alpha_4 = \frac{-vu}{2}$, $\alpha_5 = -\frac{v'(u+2tv)+2v^2}{2}$, $\alpha_6 = \frac{-u'(u+2tv)}{2}$, $\alpha_7 = \frac{wu'u+wvu+v'(1+2twu)}{2u}$, $\alpha_8 = \frac{v(1+2twu)}{2u}$, $\alpha_9 = \frac{v}{2u}$, $C^s_{ij|t}$ *is the h-covariant derivative of Cartan tensor* $C^s_{jk} := \frac{1}{2}g^{si}\frac{\partial g_{ij}}{\partial y^k}$ *with respect to the Cartan connection and* $F^k_{ij} := \frac{g^{kl}}{2}(\frac{\delta g_{jl}}{\delta x^i} + \frac{\delta g_{il}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^l})$.

Lemma 2. *Let $\beta_1, \beta_2, \beta_3, \beta_4 \in C^\infty(TM_0)$ such that $\beta_1 y_j\delta^h_i + \beta_2 y_i\delta^h_j + \beta_3 y_i y_j y^h + \beta_4 g_{ij}y^h = 0$. Then $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$.*

We denote by $FC = (HTM_0, \nabla^c)$ the Cartan connection of (M, F), where ∇^c is a linear connection on VTM_0 given by

$$\nabla^c_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = C^k_{ij}\frac{\partial}{\partial y^k}, \qquad \nabla^c_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} = F^k_{ij}\frac{\partial}{\partial y^k}. \tag{7}$$

For $X, Y \in \Gamma(TTM_0)$, the Levi-Civita connection $\widetilde{\nabla}$ induces a connection ∇ on VTM_0 defined by $\nabla_X VY = V(\widetilde{\nabla}_X VY)$ where V is the projection morphism of TTM_0 on VTM_0 [2]. Using (3) and (5), it is easy to check that ∇ has the following expression:

$$\nabla_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = \left[C^k_{ij} + \alpha_1 g_{ij}y^k - \alpha_2\left(y_i\delta^k_j + y_j\delta^k_i\right) + \alpha_3 y_i y_j y^k\right]\frac{\partial}{\partial y^k}, \tag{8}$$

$$\nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} = V\widetilde{\nabla}_{\delta_j}\dot{\partial}_i = F^k_{ij}\frac{\partial}{\partial y^k}. \tag{9}$$

Theorem 3. *Let ∇^c be the linear connection of the Cartan connection FC and ∇ be the projection of the Levi-Civita connection $\widetilde{\nabla}$ on VTM_0. Then for $X \in \Gamma(TTM_0)$, $Y \in \Gamma(VTM_0)$ the relation $\nabla_X Y = \nabla^c_X Y$ holds if and only if $u = k$ and $v = 0$, where k is a real constant.*

Proof. By (7), (8) and (9), $\nabla_X Y = \nabla^c_X Y$ if and only if $\alpha_1 g_{ij}y^k - \alpha_2(y_i\delta^k_j + y_j\delta^k_i) + \alpha_3 y_i y_j y^k = 0$. Lemma 2 implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. But these relations are hold if and only if $u' = 0$ and $\omega = 0$. □

Now, we consider the various kinds of foliation which are naturally associated to (TM_0, G) [7].

Theorem 4. *The Riemannian metric G is bundle-like for the vertical foliation $\mathcal{F}_V$ if and only if (M, F) is a Riemannian manifold and $u = k$ and $v = 0$, where k is a real constant.*

Proof. Using (2) and (4) we deduce that G is bundle-like for $\mathcal{F}_V$ if and only if

$$\left[\left(-u^2 C^s_{ij} + \alpha_4\big(y_j\delta^s_i + y_i\delta^s_j\big) + \alpha_5 y_i y_j y^s + \alpha_6 g_{ij} y^s + \frac{1}{2}R^r{}_{ij}\right) + (i, j)\right] H_{rk} = 0, \tag{10}$$

where (i, j) shows the permutation of the indices i, j and summation. Since C^r_{ij} and R^r_{ij} are symmetric and skew-symmetric with respect i, j, respectively, then (10) is equivalent to following

$$-u^2 C^s_{ij} + \alpha_4\big(y_j\delta^s_i + y_i\delta^s_j\big) + \alpha_5 y_i y_j y^s + \alpha_6 g_{ij} y^s = 0. \tag{11}$$

We show that (11) is hold if and only if $C^s_{ij} = 0$, $u = k$ and $v = 0$. If $u = k$ and $v = 0$, then $\alpha_4 = \alpha_5 = \alpha_6 = 0$. Hence if $C^s_{ij} = 0$, then we get (11). Conversely, let (11) is hold. Multiplying it with y^j, gives us $\alpha_4 F^2\delta^s_i + (\alpha_4 + \alpha_5 F^2 + \alpha_6) y_i y^s = 0$. By Lemma 2, it follows that $\alpha_4 = 0$ and $\alpha_4 + \alpha_5 F^2 + \alpha_6 = 0$. Thus $u' = 0$ and $v = 0$. Setting these relations in (11) implies that $C^s_{ij} = 0$ and M is a Riemannian manifold. □

Theorem 5. *The Finsler manifold (M, F) is a Landsberg manifold if and only if the vertical foliation $\mathcal{F}_V$ is totally geodesic on (TM_0, G).*

Proof. Since the vertical distribution is spanned by $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$, then $\mathcal{F}_V$ is totally geodesic if and only if $\widetilde{\nabla}_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} \in \Gamma(VTM_0)$ or $H\widetilde{\nabla}_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = 0$, $i, j = 1, \ldots, n$. By (3), we get $H\widetilde{\nabla}_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = \frac{1}{u^2}(G^s_{ij} - F^s_{ij})\frac{\delta}{\delta x^s} = 0$ or equivalently $G^s_{ij} = F^s_{ij}$. This means that (M, F) is a Landsberg manifold. □

Here, we consider the notation from [9] related to foliated manifolds entitled Vaisman connection. At first, note that the torsion T has the decomposition $T(X, Y) = V(T(X, Y)) + H(T(X, Y))$, where H is the projection morphism of TTM_0 on HTM_0.

Proposition 6. *Let (M, F) be a Finsler manifold. Then the Vaisman connection ∇^v on $(TM_0, \mathcal{F}_V, G)$ is locally expressed with respect to the adapted local basis $\{\delta_i, \dot{\partial}_i\}$ as follows*:

$$\nabla^v_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = \big[C^k_{ij} - \alpha_2\big(y_i\delta^k_j + y_j\delta^k_i\big) + \alpha_1 g_{ij} y^k + \alpha_3 y_i y_j y^k\big]\frac{\partial}{\partial y^k}, \tag{12}$$

$$\nabla^v_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} = G^k_{ij}\frac{\partial}{\partial y^k}, \qquad \nabla^v_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = F^k_{ij}\frac{\delta}{\delta x^k}, \qquad \nabla^v_{\frac{\partial}{\partial y^j}}\frac{\delta}{\delta x^i} = 0. \tag{13}$$

Theorem 7. *Let (M, F) be a Finsler manifold. Then the Levi-Civita and the Vaisman connections on the foliated manifold $(TM_0, \mathcal{F}_V, G)$ induce the same connection on the structural bundle if and only if (M, F) is a Landsberg manifold.*

Proof. The relations (12) and (13) give the local expression of the induced connection by the Vaisman connection on VTM_0. According to (8), (9), (12) and (13) we deduce that the Levi-Civita and the Vaisman connections on the foliated manifold $(TM_0, \mathcal{F}_V, G)$ induce the same connection on the structural bundle if and only if $G^k_{ij} = F^k_{ij}$ or equivalently M is a Landsberg manifold. □

By using Theorems 5 and 7, we have the following.

Corollary 8. *Let (M, F) be a Finsler manifold. Then the Levi-Civita connection and the Vaisman connection on the foliated manifold (TM, F, G) induce the same connection on VTM if and only if the foliation F_V is totally geodesic.*

A Riemannian foliated manifold with the Riemannian metric G is called a Reinhart manifold if and only if $(\nabla^v_X G) \times (Y, Z) = 0$, for all the sections X of the structural bundle and Y, Z sections of the transversal bundle, where the covariant derivative is taken with respect to the Vaisman connection of the manifold.

Theorem 9. *Let (M, F) be a Finsler manifold. The foliated manifold $(TM_0, \mathcal{F}_V, G)$ is a Reinhart manifold if and only if (M, F) is a Riemannian manifold.*

Proof. Let $X = X^i\frac{\partial}{\partial y^i} \in \Gamma(VTM_0)$ and $Y = Y^j\frac{\delta}{\delta x^j}$, $Z = Z^k\frac{\delta}{\delta x^k}$ belong to $\Gamma(HTM_0)$, also ∇^v be the Vaisman connection on $(TM_0, \mathcal{F}_V, G)$. By (5) and (13), we obtain

$$\begin{aligned}(\nabla^v_X G)(Y,Z) &= X^i \frac{\partial}{\partial y^i} G\left(Y^j \frac{\delta}{\delta x^j}, Z^k \frac{\delta}{\delta x^k}\right) - G\left(\nabla^v_{X^i \frac{\partial}{\partial y^i}} Y^j \frac{\delta}{\delta x^j}, Z^k \frac{\delta}{\delta x^k}\right) - G\left(Y^j \frac{\delta}{\delta x^j}, \nabla^v_{X^i \frac{\partial}{\partial y^i}} Z^k \frac{\delta}{\delta x^k}\right)\\ &= X^i \frac{\partial}{\partial y^i}\left(Y^j Z^k g_{jk}\right) - X^i \frac{\partial}{\partial y^i}\left(Y^j\right) Z^k g_{jk} - Y^j X^i \frac{\partial}{\partial y^i}\left(Z^k\right) g_{kj} = 2X^i Y^j Z^k C_{ijk}.\end{aligned}$$

Hence M is a Reinhart manifold if and only if $C_{ijk} = 0$ and then M is a Riemannian manifold. □

By Theorems 4 and 9, we see that the Riemannian metric G is bundle-like for the vertical foliation $\mathcal{F}_V$ if and only if $(TM_0, \mathcal{F}_V, G)$ is a Reinhart space and $u = k$ and $v = 0$.

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