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Partial Differential Equations

Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map

Dynamique explosive de solutions régulières équivariantes de l'application de Schrödinger énergie critique

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ABSTRACT

We consider the energy critical Schrödinger map $\partial_t u = u \wedge \Delta u$ to the 2-sphere for equivariant initial data of homotopy number $k = 1$. We show the existence of a set of smooth initial data arbitrarily close to the ground state harmonic map Q_1 in the scale invariant norm $\dot{H}^1$ which generate finite time blow up solutions. We give in addition a sharp description of the corresponding singularity formation which occurs by concentration of a universal bubble of energy

$$u(t,x) - e^{\Theta^* R} Q_1\left(\frac{x}{\lambda(t)}\right) \to u^* \quad \text{in } \dot{H}^1 \text{ as } t \to T$$

where $\Theta^* \in \mathbb{R}$, $u^* \in \dot{H}^1$, R is a rotation and the concentration rate is given for some $\kappa(u) > 0$ by

$$\lambda(t) = \kappa(u) \frac{T-t}{|\log(T-t)|^2}\big(1 + o(1)\big) \quad \text{as } t \to T.$$

RÉSUMÉ

Nous considérons l'application de Schrödinger sur la 2-sphère énergie critique $\partial_t u = u \wedge \Delta u$ pour des données initiales à symétrie équivariante et de degré $k = 1$. Nous exhibons un ensemble de données initiales régulières arbitrairement proches dans la topologie invariante d'échelle $\dot{H}^1$ de l'application harmonique d'énergie minimale Q_1 qui engendrent des solutions explosives en temps fini. Nous donnons une description fine de la formation de singularité qui correspond à la concentration d'une bulle universelle d'énergie

$$u(t,x) - e^{\Theta^* R} Q_1\left(\frac{x}{\lambda(t)}\right) \to u^* \quad \text{in } \dot{H}^1$$

où $\Theta^* \in \mathbb{R}$, $u^* \in \dot{H}^1$, R est une rotation et la vitesse de concentration est donnée pour une certain $\kappa(u) > 0$ par:

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$$\lambda(t)=\kappa(u)\frac{T-t}{|\log(T-t)|^2}\big(1+o(1)\big)\quad \text{quand } t\to T.$$

Version française abrégée

Nous considérons l'application de Schrödinger énergie critique sur la 2-sphère

$$\begin{cases}\partial_t u = u\wedge \Delta u,\\ u_{|t=0}=u_0\in \dot H^1,\end{cases}\quad (t,x)\in\mathbb{R}\times\mathbb{R}^2,\ u(t,x)\in\mathbb{S}^2. \tag{1}$$

Ce système appartient à une classe d'équations géométriques énergie critique qui inclut le flot parabolique de la chaleur harmonique cf. [21,16,17,22,3] et les applications de type onde, cf. [19], et apparaît notamment en ferromagnétisme en relation avec les équations de Landau–Lifschitz. Ce système est Hamiltonien et le flot laisse invariante l'énergie de Dirichlet

$$E\big(u(t)\big)=\int_{\mathbb{R}^2}\big|\nabla u(t,x)\big|^2\,\mathrm{d}x=E(u_0). \tag{2}$$

Nous considérons des flots à symétrie k-équivariante

$$u(t,x)=e^{k\theta R}\left|\begin{array}{l}u_1(t,r),\\ u_2(t,r),\\ u_3(t,r),\end{array}\right.\qquad R=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},$$

où (r,θ) désignent les coordonnées polaires sur $\mathbb{R}^2$, et où $k\in\mathbb{Z}^*$ est le degré de l'application. Dans ce cas, le problème de Cauchy est bien posé localement en temps dans $\dot H^1$ par [4,6,7]. Pour une donnée initiale générale le resultat est connu uniquement pour donnée petite [2].

Le minimiseur de l'énergie de Dirichlet (2) à degré fixé est explicitement donné par

$$Q_k(x)=e^{k\theta R}\left|\begin{array}{l}\frac{2r^k}{1+r^{2k}}\\ 0\\ \frac{1-r^{2k}}{1+r^{2k}}\end{array}\right.$$

et engendre une solution stationnaire de (1). Pour $k\geqslant 3$, cette solution est stable et même asymptotiquement stable d'après Gustafson, Nakanishi, Tsai [8]. Pour $k=1$, le premier résultat d'instabilité de $Q\equiv Q_1$ dans la topologie invariante d'échelle $\dot H^1$ est donné par Bejenaru et Tataru [1]. Dans la continuation des travaux sur l'équation de Schrödinger L^2 critique [13,10,11,14,12,18] et l'application d'ondes énergie critique sur la 2-sphère [19], nous obtenons le premier résultat d'explosion en temps fini :

Théorème *(Explosion pour $k=1$). Il existe un ensemble de donnée initiales régulières à symétrie équivariante, de degré $k=1$ et arbitrairement proches de Q_1 dans $\dot H^1$ telles que la solution correspondante de* (1) *explose en temps fini $T<+\infty$ par concentration d'une bulle universelle d'énergie*

$$u(t,x)-e^{\Theta^* R}Q_1\left(\frac{x}{\lambda(t)}\right)\to u^*\quad \text{dans } \dot H^1 \text{ quand } t\to T$$

où $\Theta^*\in\mathbb{R}$, $u^*\in\dot H^1$, et la vitesse de concentration est donnée pour un certain $\kappa(u)>0$ par :

$$\lambda(t)=\kappa(u)\frac{T-t}{|\log(T-t)|^2}\big(1+o(1)\big)\quad \text{quand } t\to T.$$

Cette note est une version abrégée de [15].

1. Setting of the problem and main result

In this paper we consider the energy critical Schrödinger map

$$\begin{cases}\partial_t u = u\wedge \Delta u,\\ u_{|t=0}=u_0\in \dot H^1,\end{cases}\quad (t,x)\in\mathbb{R}\times\mathbb{R}^2,\ u(t,x)\in\mathbb{S}^2. \tag{3}$$

This equation is related to the Landau–Lifschitz equation for ferromagnetism, and it belongs to a class of geometric evolution equations [21,16,17,22,3] including wave maps and the harmonic heat flow which have attracted a considerable attention in the past ten years, see [19] for a more complete introduction to this class of problems.

The Hamiltonian structure of the problem implies conservation of the Dirichlet energy

$$E\big(u(t)\big) = \int_{\mathbb{R}^2} \big|\nabla u(t,x)\big|^2 \,\mathrm{d}x = E(u_0) \tag{4}$$

which moreover is left unchanged by the scaling symmetry of the problem $u(t,x) \mapsto u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$. The question of the global existence of all solutions or on the contrary the possibility of a finite blow up and singularity formation corresponding to a concentration of energy has been addressed recently in detail for the wave map problem—the wave analogue of (3)—and the Yang–Mills equations, see [19] and references therein, [9] (see also [21,16,17,22,5,3] for the heat flow), and has been until now open for the Schrödinger map problem.

We shall focus on the case of solutions with k-equivariant symmetry

$$u(t,x) = e^{k\theta R} \left| \begin{array}{l} u_1(t,r), \\ u_2(t,r), \\ u_3(t,r), \end{array} \right. \qquad R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where (r,θ) are the polar coordinates on $\mathbb{R}^2$, and $k \in \mathbb{Z}^*$ is the homotopy number. In this case, the Cauchy problem is well-posed in $\dot H^1$ if the energy E is sufficiently small, [4] or, more generally if the energy E is sufficiently close to the minimum in a given homotopy class k, realized on a harmonic map $Q_k : \mathbb{R}^2 \to \mathbb{S}^2$, [6,7]. In the general case without symmetry only the small data result is available [2]. In a given homotopy class, the minimizer of the Dirichlet energy (4) is explicitly given by the harmonic map

$$Q_k(x) = e^{k\theta R} \left| \begin{array}{l} \frac{2r^k}{1+r^{2k}} \\ 0 \\ \frac{1-r^{2k}}{1+r^{2k}} \end{array} \right.$$

which generates a stationary solution to (3). For large degree $k \geqslant 3$, this solution is stable and in fact asymptotically stable from Gustafson, Nakanishi and Tsai [8]. For $k=1$ which corresponds to least energy maps, Bejenaru and Tataru [1] exhibit some instability mechanism of $Q \equiv Q_1$ in the scale invariant space $\dot H^1$.

In the continuation of the works on the L^2 critical nonlinear Schrödinger equation [13,10,11,14,12,18] and the sharp description of a stable blow up for the wave map problem to the 2-sphere [19], we obtain in [15] the existence of singularity formation arising from smooth data arbitrarily close to Q for $k=1$:

Theorem 1.1 *(Existence and sharp description of a blow up for $k=1$). Let $k=1$. There exists a set of smooth equivariant initial data of degree $k=1$ arbitrarily close to the ground state Q_1 in the $\dot H^1$ topology such that the corresponding solution to* (3) *blows up in finite time through the concentration of a universal bubble of energy*

$$u(t,x) - e^{\Theta^* R} Q_1\left(\frac{x}{\lambda(t)}\right) \to u^* \quad \text{in } \dot H^1 \text{ as } t \to T,$$

for some $\Theta^* \in \mathbb{R}$, and at a speed given for some $\kappa(u) > 0$ by:

$$\lambda(t) = \kappa(u) \frac{T-t}{|\log(T-t)|^2}\big(1+o(1)\big) \quad \text{as } t \to T. \tag{5}$$

We stress that the blow up speed (5) is a natural candidate for stable blow up, see [3] for the parabolic problem, and a new instability mechanism occurs for the Schrödinger map.

2. Strategy of the proof

Step 1. Choice of gauge.

We describe the flow (3) in the renormalized Frenet basis associated to the harmonic map Q_1:

$$(e_r, e_\tau, Q_1), \qquad e_r = \frac{\partial_r Q_1}{|\partial_r Q_1|}, \qquad e_\tau = \frac{\partial_\tau Q_1}{|\partial_\tau Q_1|}, \qquad \partial_\tau = \frac{1}{r}\partial_\theta.$$

We renormalize the map

$$u(t,x)=e^{\Theta(t)R}v(s,y),\qquad \frac{ds}{dt}=\frac{1}{\lambda^2},\qquad y=\frac{x}{\lambda}$$

and rewrite the equation for w in the Frenet basis:

$$v(s,y)=\alpha(s,y)e_r+\beta(s,y)e_\tau+\big(1+\gamma(s,y)\big)Q_1,\quad \alpha^2+\beta^2+(1+\gamma)^2=1.$$

To leading order, the flow near Q becomes a quasilinear Schrödinger equation:

$$i\partial_s w-\mathcal{H}w+ib\Lambda w-aw=NL(w),\quad w=\alpha+i\beta, \tag{6}$$

where we introduced the generator of the scaling symmetry $\Lambda f=y\cdot\nabla f$ and the modulation parameter

$$b=-\frac{\lambda_s}{\lambda},\qquad a=-\Theta_s,$$

and where the linearized operator is explicitly given by

$$\mathcal{H}w=-\Delta w+\frac{y^4-6y^2+1}{y^2(1+y^2)^2}.$$

Step 2. Construction of the approximate profile and formal derivation of the law.

We now proceed as in [14,20,19] and look for a suitable approximate solution to the renormalized equation (6) in the form of an homogeneous expansion

$$w_0(s,y)=\alpha_0(s,y)+i\beta_0(s,y)$$

with

$$\alpha_0=aT_{1,0}+b^2T_{0,2},\qquad \beta_0=bT_{0,1}+abT_{1,1}+b^3T_{0,3},\qquad \gamma_0=b^2S_{0,2}.$$

The key is to *look for the law of the modulation parameter $s\mapsto(a,b)$ such that at each step we may solve for a sufficiently decaying profile $T_{i,j}$*. At the order b, we get

$$\mathcal{H}T_{0,1}=\Lambda\phi,\qquad \phi(y)=2\tan^{-1}\left(\frac{1}{y}\right)$$

which yields a growing solution for y large $T_{0,1}(y)\sim y|\log y|-y$ as $y\to+\infty$. An explicit computation then reveals that the choice to leading order

$$b_s\sim -b^2-a^2,\qquad a_s\sim 0 \tag{7}$$

is the unique choice which allows us to solve the $T_{i,j}$ system with enough decay as $1\ll y$. In fact, exactly like in [14,19], a *flux computation* based on the slow decay of the radiative terms $T_{i,j}$ allows one to compute the additional logarithmic corrections induced by non-trivial boundary terms at infinity:

$$b_s+b^2\sim -\frac{b^2}{2|\log b|}-a^2,\qquad a_s\sim -2\frac{ab}{|\log b|}. \tag{8}$$

A new phenomenon here is that the acceleration of the phase a acts as *a damping force against concentration* through the b equation (8), and the dynamical system (8) admits a one-dimensional set of initial data for which: $|a|\ll\frac{b}{|\log b|}$. The integration of the modulation equation in this regime:

$$b_s+b^2=-\frac{b^2}{2|\log b|},\quad |a|\ll\frac{b}{|\log b|},\qquad b=-\frac{\lambda_s}{\lambda},\qquad \frac{ds}{dt}=\frac{1}{\lambda^2},\qquad \Theta_s=-a, \tag{9}$$

now yields finite time blow up $\lambda(t)\to 0$ as $t\to T$ for some finite $T<+\infty$ together with the asymptotics near blow up time (5) and the convergence $\Theta(t)\to\Theta^*$ as $t\to T$.

Step 3. Control of the remainder: the mixed energy/Morawetz Lyapunov functional.

After the approximate solution is constructed, we use modulation theory to introduce a suitable nonlinear decomposition of the flow

$$u(t,x)=e^{\Theta(t)R}\big[(\alpha_0+\alpha)(s,r)e_r+(\beta_0+\beta)(s,y)e_\tau+(\gamma_0+\gamma)Q\big](s,y)$$

where the four modulation parameters (λ, b, Θ, a) are chosen, by a standard modulation argument, to ensure that $w = \alpha + i\beta$ is orthogonal to the kernel of $\mathcal{H}^2$. Recall that $\mathcal{H}$ is a positive operator with a resonance $\mathcal{H}(\Lambda\phi) = 0$ generated by the scaling and phase invariances. Our strategy to control the reminder term w, similar to [19], is to exhibit a Lyapunov functional which mixes an energy and Morawetz type identities. The are three main differences with the analysis in [19]. First we need to take more derivatives of the equation to overcome the growth of the radiation, and the Schrödinger map problem is in some sense two derivatives above the wave map problem. Second, the *quasilinear* structure of the problem needs to be addressed through the use of suitable derivatives which are compatible with the geometry of the system. Third we need to construct the codimension one set of initial data to excite the suitable solution to (8). Using in particular factorization properties of $\mathcal{H}$, our main tool is the construction of a Lyapunov type functional at the Sobolev H^4 level which, thanks to the construction of a sufficiently high order approximate profile and the *four* orthogonality conditions on w, yields a uniform bound:

$$\|w\|_{H^4}^2 \lesssim \left\|\mathcal{H}^2 w\right\|_{L^2}^2 \lesssim \frac{b^4}{|\log b|^2} \tag{10}$$

which is sufficient to verify the modulation equations (9).

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