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Probability Theory

On the convergence of orthogonal series

Sur la convergence des systèmes orthogonaux

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ABSTRACT

In this Note we present a new approach to the complete characterization of the a.s. convergence of orthogonal series. We sketch a new proof that a.s. convergence of $\sum_{n=1}^{\infty} a_n\varphi_n$ for all orthonormal systems $(\varphi_n)_{n=1}^{\infty}$ is equivalent to the existence of a majorizing measure on the set $T = \{\sum_{m=n}^{\infty} a_m^2\colon\ n \geqslant 1\} \cup \{0\}$. The method is based on the chaining argument used for a certain partitioning scheme.

RÉSUMÉ

Nous proposons une nouvelle approche pour démontrer que la convergence presque sure de la série $\sum_{n=1}^{\infty} a_n\varphi_n$ pour tous les systèmes orthogonaux $(\varphi_n)_{n=1}^{\infty}$ est équivalente à l'existence d'une mesure majorante sur l'ensemble $T = \{\sum_{m=n}^{\infty} a_m^2\colon\ n \geqslant 1\} \cup \{0\}$. L'ingrédient principal est une nouvelle méthode de construction de séries orthogonales.

1. Introduction

An orthonormal sequence $(\varphi_n)_{n=1}^{\infty}$ on a probability space $(\Omega, \mathbf{F}, \mathbf{P})$ is a sequence of random variables $\varphi_n : \Omega \to \mathbb{R}$ such that $\mathbf{E}\varphi_n^2 = 1$ and $\mathbf{E}\varphi_n\varphi_m = 0$ whenever $n \neq m$. The problem we treat in this Note is how to characterize the sequences of $(a_n)_{n=1}^{\infty}$ for which the series

$$\sum_{n=1}^{\infty} a_n\varphi_n \quad \text{converges a.e. for any orthonormal } (\varphi_n)_{n=1}^{\infty}, \tag{1}$$

on all probability spaces $(\Omega, \mathbf{F}, \mathbf{P})$. Note that we can assume $a_n > 0$, for $n \geqslant 1$. It occurs that the answer is related to the analysis of the set

$$T = \left\{ \sum_{m=1}^{n} a_m^2\colon\ n \geqslant 1 \right\} \cup \{0\}.$$

A trivial observation is that to have the series convergent one needs T to be compact.

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The characterization should be stated in terms of geometry of T. There were several steps towards the general result. For various applications it suffices to use the Rademacher–Menchov theorem (see [4]).

Theorem 1. *Whenever*

$$\sum_{n=1}^{\infty} a_n^2 \log^2(n+1) < \infty,$$

then for each orthonormal sequence $(\varphi_n)_{n=1}^{\infty}$ the series $\sum_{n=1}^{\infty} a_n\varphi_n$ is a.e. convergent.

A more involved analysis is based on the study of regular partitions of T. Suppose that $T \subset [0, M)$, then define $\mathbf{A}_k = \{A_i^{(k)}\colon\ 0 \leqslant i < 4^k\}$, $k \geqslant 0$ where $A_i^{(k)} = [i4^{-k}M, (i+1)4^{-k}M) \cap T$. Let $N_k = \{i \in \{0, 1, \ldots, 4^{k-1}\}\colon\ A_i^{(k)} \neq \emptyset\}$ and $T_k = \bigcup_{i\in N_k}[i4^{-k}M, (i+1)4^{-k}M)$. By $\|\cdot\|$ denote the L_2-norm on $L_2(0,1)$. It is proved in [6] (see also [7]) that there exists a permutation σ on $\mathbb{N}$ for which $\sum_{n=1}^{\infty} a_{\sigma(n)}\varphi_n$ converges a.e. for any orthonormal $(\varphi_n)_{n=1}^{\infty}$ if and only if $\|\sum_{k=1}^{\infty} 1_{T_k}\| < \infty$. Moreover (see [12] and [7]) $\sum_{n=1}^{\infty} a_{\sigma(n)}\varphi_n$ converges for all permutations σ on $\mathbb{N}$ and orthonormal $(\varphi_n)_{n=1}^{\infty}$ if and only if $\sum_{k=1}^{\infty} \|1_{T_k}\| < \infty$.

The complete characterization of (1) was finally presented in [7,8]. The approach is based on a deep study of partitions $\mathbf{A}_k$, $k \geqslant 0$ and the following classical result of Tandori [12]:

Theorem 2. *For each orthonormal sequence $(\varphi_n)_{n=1}^{\infty}$ the series $\sum_{n=1}^{\infty} a_n\varphi_n$ converges a.e. if and only if*

$$\mathbf{E}\sup_{m\geqslant 1}\left(\sum_{n=1}^{m} a_n\varphi_n\right)^2 < \infty.$$

Several equivalent conditions characterizing (1) are given in [8]. For our purposes we choose the language of majorizing measures. Let

$$d(s,t) = \sqrt{|s-t|}, \quad s,t \in T, \qquad B(t,\varepsilon) = \big\{s \in T\colon\ d(s,t) \leqslant \varepsilon\big\}.$$

A Borel probability measure μ on T is called majorizing (in the orthogonal setting) if

$$\sup_{t\in T}\int_0^M \big(\mu\big(B(t,\varepsilon)\big)\big)^{-\frac{1}{2}}\,\mathrm{d}\varepsilon < \infty.$$

Theorem 3. *The series* (1) *converges for all orthonormal $(\varphi_n)_{n=1}^{\infty}$ if and only if there exists a majorizing measure on T.*

2. Majorizing measures in the orthogonal setting

Majorizing measures were invented to characterize sample boundedness for certain stochastic processes. The simplest way to control a process $X(t)$, $t \in T$ is to consider all its increments $X(t) - X(s)$, $s,t \in T$. We say that a process $X(t)$, $t \in T$ is of suborthogonal increments if

$$\mathbf{E}\big(X(t) - X(s)\big)^2 \leqslant d(s,t)^2, \quad s,t \in T. \tag{2}$$

Under the increment condition the existence of a majorizing measure implies sample boundedness. The result was first proved in [9] and generalized in [1]. By Theorem 3.2 in [1]:

Theorem 4. *If there exists a majorizing measure m on T, then for each process $X(t)$, $t \in T$ that satisfies* (2) *the following inequality holds*:

$$\mathbf{E}\sup_{s,t\in T}\big(X(t) - X(s)\big)^2 \leqslant 16\cdot 5^{\frac{5}{2}}\left(\sup_{t\in T}\int_0^M \big(\mu\big(B(t,\varepsilon)\big)\big)^{-\frac{1}{2}}\,\mathrm{d}\varepsilon\right)^2 < \infty.$$

The difficult part is to give a complete characterization of sample boundedness for a certain process or a class of processes. The first example [3] (cf. [10]) which validated the majorizing measure definition was that for any ultrametric space the existence of a majorizing measure is a sufficient and necessary condition for all processes of bounded increments to be sample bounded. Then appeared the characterization of sample boundedness for Gaussian processes [9] and many other canonical processes [11,5]. Also, the author could generalize the result for the ultrametric spaces to a setting [2] which in the special suborthogonal case gives:

Theorem 5. *Whenever each process $X(t)$, $t \in T$ that satisfies* (2) *is sample bounded then there exists a majorizing measure on T.*

Consequently Theorems 4 and 5 imply that the sample boundedness of all suborthogonal processes on T is equivalent to the existence of a majorizing measure. The proof of Theorem 5 is based on Fernique's [3] (see also [10]) technique of constructing a majorizing measure.

Theorem 6. *Whenever each probability Borel measure μ on T is weakly majorizing i.e.*

$$\sup_{\mu} \int_T \int_0^M \big(\mu\big(B(t,\varepsilon)\big)\big)^{-\frac{1}{2}}\, \mathrm{d}\varepsilon\, \mu(\mathrm{d}t) < \infty$$

then there exists a majorizing measure on T.

Now we turn to the main question of characterizing (1). We say that a process $X(t)$, $t \in T$ has orthogonal increments if

$$\mathbf{E}\big(X(t) - X(s)\big)^2 = d(s,t)^2, \quad s, t \in T. \tag{3}$$

Recall that $T = \{\sum_{m=1}^n a_m^2\colon\ n \geqslant 1\} \cup \{0\}$. There is a bijection between orthonormal series $\sum_{n=1}^\infty a_n \varphi_n$ and processes with orthogonal increments on T. Namely for each orthonormal sequence $(\varphi_n)_{n=1}^\infty$ we define the process $X(t) = \sum_{n=1}^m a_n \varphi_n$, for $t = \sum_{n=1}^m a_n^2$, $X(0) = 0$ and for each process $X(t)$, $t \in T$ we define orthonormal $\varphi_m = a_m^{-1}(X(\sum_{n=1}^m a_n^2) - X(\sum_{n=0}^{m-1} a_n^2))$, $m > 1$ and $\varphi_1 = X(a_1^2) - X(0)$. Therefore by Theorem 2 each orthogonal series $\sum_{n=1}^\infty a_n \varphi_n$ is a.e. convergent if and only if there exists a universal constant $\mathbf{M} < \infty$ such that

$$\mathbf{E} \sup_{t \in T} \big|X(t) - X(0)\big|^2 \leqslant \mathbf{M} \tag{4}$$

for all orthogonal processes on T. This class of processes is significantly smaller than the class of suborthogonal processes. Our main result is the following:

Theorem 7. *If all orthogonal processes satisfy* (4) *then*

$$\sup_{\mu} \int_T \int_0^M \big(\mu\big(B(t,\varepsilon)\big)\big)^{-\frac{1}{2}} \leqslant M < \infty.$$

Together with Theorems 4, 5, 6 this completes a new proof of Theorem 3. The proof of Theorem 7 is based on the study of natural partitions $\mathbf{A}_k$, $k \geqslant 0$ and a special partitioning scheme.

3. Regular partitions

We start the analysis translating the language of weakly majorizing measures into the language of natural partitions $\mathbf{A}_k$, $k \geqslant 0$. Note that if $t \in A_i^{(k)}$ then $A_i^{(k)} \subset B(t, 2^{-k}M)$, and therefore

$$\int_T \big(\mu\big(B\big(t, 2^{-k}M\big)\big)\big)^{-\frac{1}{2}} \mu(\mathrm{d}t) \leqslant \sum_{i=0}^{4^k-1} \int_{A_i^{(k)}} \big(\mu\big(A_i^{(k)}\big)\big)^{-\frac{1}{2}} \mu(\mathrm{d}t) \leqslant \sum_{i=0}^{4^k-1} \big(\mu\big(A_i^{(k)}\big)\big)^{\frac{1}{2}}.$$

Consequently

$$\int_T \int_0^M \big(\mu\big(B(t,\varepsilon)\big)\big)^{-\frac{1}{2}}\, \mathrm{d}\varepsilon\, \mu(\mathrm{d}t) \leqslant M \sum_{k=1}^\infty 2^{-k} \sum_{i=0}^{4^k-1} \big(\mu\big(A_i^{(k)}\big)\big)^{\frac{1}{2}}.$$

The second point is that given μ not all subsets $A_i^{(k)} \in \mathbf{A}_k$ are important. Let $1 < c < 2 < C$. We define $I^{(k)}$ as the set of indexes $i \in \{0, 1, \ldots, 4^k - 1\}$ for which $A_i^{(k)} \neq \emptyset$ and

$$C^{-1}\mu\big(A_{[i/4]}^{(k-1)}\big) \leqslant \mu\big(A_i^{(k)}\big) \leqslant c^{-1}\mu\big(A_{4[i/4]}^{(k)} \cup A_{4[i/4]+2}^{(k)}\big) \leqslant c^{-1}\mu\big(A_{[i/4]}^{k-1}\big), \quad \text{if } 2 \mid i,$$
$$C^{-1}\mu\big(A_{[i/4]}^{(k-1)}\big) \leqslant \mu\big(A_i^{(k)}\big) \leqslant c^{-1}\mu\big(A_{4[i/4]+1}^{(k)} \cup A_{4[i/4]+3}^{(k)}\big) \leqslant c^{-1}\mu\big(A_{[i/4]}^{k-1}\big), \quad \text{if } 2 \nmid i.$$

The main observation is that to show that μ is weakly majorizing one need only care about $A_i^{(k)}$, $i \in I^{(k)}$.

Proposition 8. *There exist* $1 < c < 2 < C$ *such that for each probability Borel measure* μ *on* T *the following inequality holds*:

$$\int_T \int_0^M \left(\mu\left(B(t,\varepsilon)\right)\right)^{-\frac{1}{2}} \leqslant L\left[1 + \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} \left(\mu\left(A_i^{(k)}\right)\right)^{\frac{1}{2}} 1_{i\in I^{(k)}}\right],$$

where $L < \infty$ *is a universal constant.*

4. The partitioning scheme

We follow an idea of Talagrand [9] of considering suitable set functionals. We define the set functionals $F_k : \mathbf{A}_k \to \mathbb{R}$, $k \geqslant 0$ by

$$F_k\left(A_i^{(k)}\right) = \sup_Y \mathbf{E} \sup_{t\in A_i^{(k)}} Y(t), \quad \text{for } 0 \leqslant i < 4^k$$

where the supremum is taken over all process $Y(t)$, $t \in \bar{A}_i^{(k)}$ (where $\bar{A}_i^{(k)} = A_i^{(k)} \cup \{i4^{-k}M, (i+1)4^{-k}M\}$), such that $\mathbf{E}Y(t) = 0$ for all $t \in \bar{A}_i^{(k)}$ and

$$\mathbf{E}\left(Y(t) - Y(s)\right)^2 = |s-t|\left(1 - 4^k M^{-1}|s-t|\right), \quad \text{for all } s, t \in \bar{A}_i^{(k)}.$$

A trivial observation is that (3) implies $F_0(T) < \infty$. The partitioning scheme is based on the following induction step:

Proposition 9. *There exists a universal constant* $K < \infty$ *such that for each* $A_i^{(k-1)} \in \mathbf{A}_{k-1}$, $k \geqslant 1$, $0 \leqslant i < 4^{k-1}$ *the following inequality holds*:

$$\left(\mu\left(A_i^{(k-1)}\right)\right)^{\frac{1}{2}} F_{k-1}\left(A_i^{(k-1)}\right) \geqslant \frac{1}{K} 2^{-k} \sum_{j=0}^{3} \left(\mu\left(A_{4i+j}^{(k)}\right)\right)^{\frac{1}{2}} 1_{4i+j\in I^{(k)}} + \sum_{j=0}^{3} \left(\mu\left(A_{4i+j}^{(k)}\right)\right)^{\frac{1}{2}} F_k\left(A_{4i+j}^{(k)}\right).$$

Since $F_0(T) < \infty$ Proposition 9 implies

$$\sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} \left(\mu\left(A_i^{(k)}\right)\right)^{\frac{1}{2}} 1_{i\in I^{(k)}} \leqslant K F_0(T) < \infty$$

and therefore Theorem 7 follows from Proposition 8.

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