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On Hardy inequalities with singularities on the boundary

Sur les inégalités de Hardy avec des singularités sur la frontière

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ABSTRACT

In this Note we present some Hardy–Poincaré inequalities with one singularity localized on the boundary of a smooth domain. Then, we consider conical domains in dimension $N \geqslant 3$ whose vertex is on the singularity and we show upper and lower bounds for the corresponding optimal constants in the Hardy inequality. In particular, we prove the asymptotic behavior of the optimal constant when the amplitude of the cone tends to zero.

RÉSUMÉ

Dans ce travail nous présentons quelques inégalités de Hardy–Poincaré avec une singularité localisée sur la frontière d'un domaine régulier. Ensuite, nous considérons des domaines coniques en dimension $N \geqslant 3$ dont le sommet est sur la singularité et nous établissons des bornes supérieure et inférieure pour les constantes optimales correspondantes dans l'inégalité de Hardy. En particulier, nous montrons le comportement asymptotique de la constante optimale lorsque l'amplitude du cône tend vers zéro.

Version française abrégée

Dans cette Note, nous nous intéressons à deux problèmes. Dans la première partie, nous donnons quelques techniques alternatives pour montrer des résultats présentés dans [12–14] portant sur les inégalités de Hardy–Poincaré dans des domaines réguliers. Dans la deuxième partie, nous traitons le problème des inégalités de Hardy optimales dans des domaines coniques en dimension $N \geqslant 3$. Une partie de ce travail a été introduite dans [7] et [6]. Peu de temps après, les preprints [12–14] ont été soumis pour publication, pendant que la version étendue du travail était en préparation. Dû au fait que nous avons obtenu des résultats similaires dans le cas des domaines réguliers, nous présentons ici brièvement uniquement les aspects originaux par rapport aux techniques développées dans [12–14].

Pour conclure, nous rappelons les résultats les plus significatifs de ce travail. Dans la section 2, nous montrons des inégalités de Hardy–Poincaré dans des domaines bornés réguliers au travers des théorèmes 2.2, 2.3, 2.4. Ces résultats se révèlent être liés à l'ellipticité de Ω à l'origine, mais aussi à la géométrie globale de Ω. Lorsque Ω n'est pas elliptique à l'origine, nous montrons une inégalité de Hardy plus faible (voir le théorème 2.5) dont la preuve requiert la dépendence continue de la constante de Hardy dans des cônes. Ce dernier résultat est présenté dans le théorème 3.1 de la section 3. En

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outre, dans la section 3, nous établissons des bornes supérieure et inférieure pour la constante optimale $\mu(\Omega)$ lorsque Ω est un cône en dimension $N \geqslant 3$ dont le sommet est situé à $x = 0$. En particulier, nous montrons la valeur asymptotique de $\mu(\Omega)$ lorsque l'amplitude du cône tend vers zéro (voir la remarque 5).

1. Introduction

Hardy inequalities represent a classical subject that has been studied intensively in the recent past, mainly motivated by its applications to PDE's involving singular potentials like $1/|x|^2$. Inverse square potentials are interesting because of their criticality since they are homogeneous of degree -2. They play a crucial role in quantum mechanics and arise in combustion models or molecular physics [8]. In [10], G.H. Hardy proved that, in the 1-d case, the optimal inequality $\int_0^\infty |v'(r)|^2\,\mathrm{d}r \geqslant 1/4 \int_0^\infty v^2(r)/r^2\,\mathrm{d}r$, holds for functions in $H_0^1(0,\infty)$. The classical multi-dimensional Hardy inequality (cf. [11]) asserts that for any Ω an open subset of $\mathbb{R}^N$, $N \geqslant 3$, it holds that

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x \geqslant \frac{(N-2)^2}{4} \int_\Omega \frac{v^2}{|x|^2}\,\mathrm{d}x, \tag{1}$$

for all $v \in H_0^1(\Omega)$. Moreover, if Ω contains the origin, the constant $(N-2)^2/4$ is optimal and it is not attained. For $N = 2$, inequality (1) is trivially true. Recently, one of the most significant improved versions of (1) for bounded domains Ω, have been established in [3,16,1].

However, Hardy inequalities with one singular potential, in which the singularity lies on the boundary have been less investigated. This Note is mainly devoted to analyze this issue. To be more precise, throughout this Note, we consider Ω to be a subset of $\mathbb{R}^N$ with the origin $x = 0$ placed on its boundary $\partial\Omega$, where the singularity is located. We then define $\mu(\Omega)$ as the best constant in the inequality $\int_\Omega |\nabla v|^2\,\mathrm{d}x \geqslant \mu(\Omega)\int_\Omega v^2/|x|^2\,\mathrm{d}x$, i.e. $\mu(\Omega) := \inf\{\int_\Omega |\nabla v|^2\,\mathrm{d}x / \int_\Omega v^2/|x|^2\,\mathrm{d}x,\ v \in H_0^1(\Omega)\}$. Of course, in view of (1), $\mu(\Omega) \geqslant (N-2)^2/4$. The authors in [13] showed that the strict inequality $\mu(\Omega) > (N-2)^2/4$ holds true when Ω is a bounded domain of class C^2. Actually, the value $\mu(\Omega)$ depends on the geometric properties of the boundary $\partial\Omega$ at the singularity. The first explicit case has been given for $\Omega = \mathbb{R}^N_+$, where $\mathbb{R}^N_+$ is the half-space of $\mathbb{R}^N$ in which the condition $x_N > 0$ holds. More precisely, for any $N \geqslant 1$, Filippas, Tertikas and Tidblom proved in [9] the new Hardy inequality:

$$\int_{\mathbb{R}^N_+} |\nabla v|^2\,\mathrm{d}x \geqslant \frac{N^2}{4} \int_{\mathbb{R}^N_+} \frac{v^2}{|x|^2}\,\mathrm{d}x, \quad \forall v \in H_0^1\big(\mathbb{R}^N_+\big). \tag{2}$$

Moreover, they proved the constant $N^2/4$ to be optimal (cf. Corollary 2.4, p. 12, [9]), i.e. $\mu(\mathbb{R}^N_+) = N^2/4$.

As a direct consequence of this result, it holds that $\mu(\Omega) \geqslant N^2/4$ for any domain Ω of class C^2 with the support in the half-space $\mathbb{R}^N_+$. In fact, for a such domain Ω, it is easy to show that $\mu(\Omega) = N^2/4$. Moreover, if Ω is a bounded domain contained in the half-space, improved Hardy–Poincaré inequalities holds (see e.g. [13]).

Another interesting situation appears in non-smooth domains Ω, when the boundary develops corners or cusps at the singularity. The most relevant example of such a domain is represented by a cone with the vertex at the origin $x = 0$. The question of studying the exact value of $\mu(\Omega)$ in cones has been full-filled in 2-d case. More precisely, if $\mathcal{C}_\gamma$ is the conical sector with the amplitude $\gamma \in (0, 2\pi)$, then (cf. [4]) $\mu(\mathcal{C}_\gamma) = \pi^2/\gamma^2$. To our knowledge, for higher dimensions $N \geqslant 3$, the value $\mu(\Omega)$ is only known when the cone Ω coincides with the half-space $\mathbb{R}^N_+$.

2. Inequalities in smooth domains

As we said above, the value of $\mu(\Omega)$ depends on the various geometric properties of Ω. In this section we assume Ω to be a Lipschitz domain with smooth boundary around the origin. Then $\partial\Omega$ is an $(N-1)$-Riemannian submanifold of $\mathbb{R}^N$ and assume that $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}$ are the principal curvatures of $\partial\Omega$ at 0. Then, up to a rotation the boundary near the origin can be written as $x_N = h(x') = \sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2)$ as $|x'| \to 0$, where $x' = (x_1, \ldots, x_{N-1}, 0)$. If we choose $\gamma < \min\{\alpha_i\colon 1 \leqslant i \leqslant N-1\}$, then $x_N > \gamma|x'|^2$ in Ω very close to origin. It means that $\Omega \cap \vartheta$, for some neighborhood of the origin ϑ, lay in the paraboloid P_γ defined by $P_\gamma = \{x = (x', x_N) \in \mathbb{R}^N \mid x_N > \gamma|x'|^2\}$. Due to the considerations above, next we distinguish four main situations.

C1. **The elliptic case:** There exists $\gamma > 0$ such that $\Omega \subset P_\gamma$. In other words, Ω lies in the corresponding elliptic paraboloid P_γ (see Fig. 1, top left).
C2. **The cylindrical case:** $\Omega \subset P_0$, where $P_0 = \mathbb{R}^N_+$ (see Fig. 1, top right).

Remark 1. In cases C1 and C2, Ω lies in $\mathbb{R}^N_+$, so, the results that are true in C2 are also valid in C1. However, we analyze them separately because we present two different tools to treat each of them.

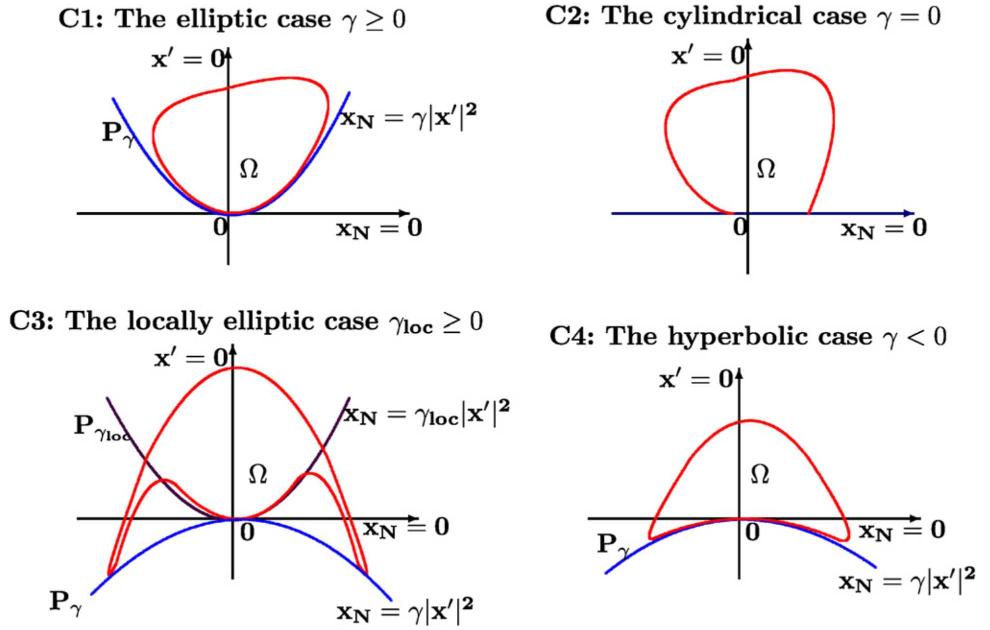

Fig. 1. The domain Ω is delimited by the interior of the red curves.

C3. **The locally elliptic case:** In this case, Ω does not lie in $\mathbb{R}^N_+$, but this happens near the origin. More precisely, we suppose that $\Omega \subset P_{\gamma_{\mathrm{loc}}}$ near the origin for some $\gamma_{\mathrm{loc}} \geqslant 0$. Away from the origin we assume $\Omega \subset P_\gamma$ for some $\gamma < 0$ (see Fig. 1, bottom left).

C4. **The hyperbolic case:** In this situation Ω has a hyperbolic geometry near the origin $x = 0$. Therefore, we suppose that $\Omega \subset P_\gamma$ for some negative $\gamma < 0$ (see Fig. 1, bottom right).

In the sequel we need the following technical lemma whose proof is based on integration by parts:

Lemma 2.1. *Let $\Omega \subset P_\gamma$ be a domain fulfilling one of the conditions* C1–C4 *in Fig.* 1, *for some constant $\gamma \in \mathbb{R}$. Given $N \geqslant 2$ and $v \in H^1_0(\Omega)$, for any constant $C \in \mathbb{R}$, the function u defined by $u(x) = v(x)|x|^C/(x_N - \gamma|x'|^2)$, fulfills the following identity*:

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x = \int_\Omega \left(x_N - \gamma\left|x'\right|^2\right)^2 |x|^{-2C}|\nabla u|^2\,\mathrm{d}x + \left(CN - C^2\right)\int_\Omega \frac{v^2}{|x|^2}\,\mathrm{d}x$$
$$+ 2\gamma\int_\Omega \left((N-1)|x|^2 - C\left|x'\right|^2\right)\left(x_N - \gamma\left|x'\right|^2\right)|x|^{-2C-2}u^2\,\mathrm{d}x. \tag{3}$$

Next we claim

Theorem 2.2. *Let $N \geqslant 3$. Assume that Ω satisfies the condition* C1 *as in Fig.* 1 (*top, left*). *Then, for all $v \in H^1_0(\Omega)$ there exists a positive constant $C(\Omega, \gamma)$ such that*

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x - \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}\,\mathrm{d}x \geqslant C(\Omega, \gamma)\int_\Omega \frac{v^2}{x_N - \gamma|x'|^2}\,\mathrm{d}x. \tag{4}$$

Sketch of the proof. We choose $C = N/2$ in the identity (3) of Lemma 2.1, taking into account that $\max_{C\in\mathbb{R}}\{CN - C^2\} = N^2/4$. Then, by easy computations we get the inequality (4). $\square$

Remark 2. Lemma 2.1 does not provide sufficient information for $\gamma = 0$. However, using spherical harmonics decomposition, we can extend and improve the result of Theorem 2.2 to the case $\gamma \geqslant 0$ as follows.

Theorem 2.3. *Let $N \geqslant 2$, and $\Omega \subset \mathbb{R}^N$ be such that the condition* C2 *is satisfied in Fig.* 1 (*top, right*). *If L is a positive number such that $L \geqslant \sup_{x\in\overline{\Omega}}|x|$, then for any $v \in H^1_0(\Omega)$,*

$$\int_\Omega |\nabla v|^2\,dx \geqslant \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}\,dx + \frac{1}{4}\int_\Omega \frac{v^2}{|x|^2\log^2(L/|x|)}\,dx. \tag{5}$$

Sketch of the proof. We fix $R > 0$ such that $\Omega \subset B_R^+$ where B_R^+ is the upper half ball of radius R, along the x_N-axis. Without losing the generality, by a density argument we may chose $v \in C_0^1(B_R^+)$. If u is the odd extension of v with respect to the component x_N, we still have $u \in C_0^1(B_R)$. We remark that u may be written in spherical harmonics as $u(x) = u(r,\sigma) = \sum_{k=1}^\infty u_k(r) f_k(\sigma)$. Here $(f_k)_{k\geqslant 0}$ is an orthonormal basis of $L^2(S^{N-1})$ constituted by the eigenvectors of the spherical Laplacian $\Delta_{S^{N-1}}$ with the corresponding eigenvalues $c_k = k(N+k-2)$, $k \geqslant 0$, and S^{N-1} is the unit sphere. We complete the proof combining the representation of the Laplace operator in spherical coordinates and a 2-d logarithmic Hardy inequality as in [1], taking advantage of the fact that the first spherical mode of u in the spherical harmonics decomposition vanishes. □

Remark 3. We emphasize that, very recently, a similar result to Theorem 2.3 has been obtained independently in [14], using a different technique which applies the so-called Emden–Fowler transform.

Theorem 2.4. *Let $N \geqslant 2$ and Ω be a domain satisfying the case* C3 *as in Fig.* 1 (*bottom, left*). *Then, for any $v \in H_0^1(\Omega)$, there exists a constant $C(\Omega)$ such that*

$$C(\Omega)\int_\Omega v^2\,dx + \int_\Omega |\nabla v|^2\,dx \geqslant \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}\,dx + \frac{1}{4}\int_\Omega \frac{v^2}{|x|^2\log^2(L/|x|)}\,dx, \tag{6}$$

where $L \geqslant \sup_{x\in\overline{\Omega}} |x|$. Note that one cannot discard the term corresponding to the L^2-norm on the left hand side because of Proposition 1 *below.*

Sketch of the proof. We apply a standard cut-off argument. Firstly, near the singularity we apply the inequality (5). Secondly, away from the origin, the potential $1/|x|^2$ is bounded. Gluing these two remarks the proof of (6) ends. For more details of a cut-off technique see e.g. [16], p. 111. □

Theorem 2.5. *Let $N \geqslant 2$ and assume that Ω satisfies the condition* C4 *as in Fig.* 1 (*bottom, right*). *For any $\epsilon > 0$, $\epsilon \ll 1$, there exists a constant $C(\Omega,\epsilon)$ such that the following inequality holds*:

$$C(\Omega,\epsilon)\int_\Omega v^2\,dx + \int_\Omega |\nabla v|^2\,dx \geqslant \left(\frac{N^2}{4} - \epsilon\right)\int_\Omega \frac{v^2}{|x|^2}\,dx, \quad \forall v \in H_0^1(\Omega). \tag{7}$$

Sketch of the proof. The proof applies local approximations of Ω around the origin, by conical sectors that cover the hyperplane $\gamma = 0$ from below. Then cut-off techniques as in theorem above end the proof. □

Remark 4. Very recently, the authors in [12] have shown that when passing to the limit $\epsilon \to 0$ in (7) the constant $C(\Omega,\epsilon)$ is uniformly bounded with respect to ϵ.

Proposition 1. *There exist smooth bounded open sets $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, satisfying either* C3 *or* C4 *in Fig.* 1, *such that $\mu(\Omega) < N^2/4$.*

Sketch of the proof. We use the fact that there are conical domains for which $\mu(\mathcal{C}) < N^2/4$. Then by the definition of $\mu(\mathcal{C})$ we can find a proper domain Ω stated above, satisfying the same inequality. □

3. Inequalities in cones

Firstly, let us consider a Lipschitz connected cone $\mathcal{C} \subset \mathbb{R}^N \setminus \{0\}$ with the vertex at zero. Let $D \subset S^{N-1}$ be the Lipschitz domain such that $\mathcal{C} = \{(r,\omega) \mid r \in (0,\infty),\ \omega \in \mathcal{D}\}$. Let $\mu(\mathcal{C})$ be the best constant in the Hardy inequality. Then $\mu(\mathcal{C}) = (N-2)^2/4 + \lambda_1(\mathcal{D})$, where $\lambda_1(\mathcal{D})$ is the Dirichlet principal eigenvalue of the spherical Laplacian $-\Delta_{S^{N-1}}$ on $\mathcal{D}$. In 2-d it is well known that (e.g. [4]) $\lambda_1(\gamma) := \lambda_1(0,\gamma) = \pi^2/\gamma^2$, where γ is the amplitude of the conical sector $\mathcal{C}_\gamma = \{(r,\omega) \mid r \in (0,\infty),\ \omega \in (0,\gamma)\}$. In higher dimensions $N \geqslant 3$, to the best of our knowledge, $\lambda_1(\mathcal{D})$ is well known only in the case where $\mathcal{D}$ is the semi-sphere S_+^{N-1} mapped in the upper half space $\mathbb{R}_+^N$. More precisely, $\lambda_1(S_+^{N-1}) = N-1$. The half space $\mathbb{R}_+^N$ corresponds to the conical sector of amplitude $\gamma = \pi/2$. The aim of this section is devoted to find lower bounds for $\lambda_1(\mathcal{D})$ in higher dimensions $N \geqslant 3$. In that sense, the definition of a cone in polar coordinates will be used.

The $N-d$ case $N \geqslant 3$. For $0 < \gamma < \pi$ we define the N-dimensional cone, with amplitude γ, denoted by $\mathcal{C}_\gamma$, consisting in all $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ such that, in spherical coordinates (cf. [15], p. 293), $\mathcal{C}_\gamma$: is given by: $x_1 =$

$r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\cos_{N-1}$, $x_2 = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\sin_{N-1},\dots,x_{N-1} = r\sin\theta_1\sin\theta_2$, $x_N = r\cos\theta_1$, where $r > 0$, $0 < \theta_1 \leqslant \gamma$, $0 \leqslant \theta_i \leqslant \pi$, $0 \leqslant \theta_{N-1} \leqslant 2\pi$, for $2 \leqslant i \leqslant N-2$. For simplicity we denote by $\lambda_1(\gamma) := \lambda_1(D_\gamma)$ the first Dirichlet eigenvalue of the spherical Laplacian on $\mathcal{D}_\gamma := \mathcal{C}_\gamma \cap S^{N-1}$. We remark that $\mu(\mathcal{C}_\gamma) = (N-2)^2/4 + \lambda_1(\gamma)$.

Theorem 3.1. *Assume that $N \geqslant 3$.*

(a) *If $0 < \gamma \leqslant \frac{\pi}{2}$ then $\lambda_1(\gamma) \geqslant \frac{(N-1)\pi^2}{4\gamma^2}$.*
(b) *For any $\epsilon > 0$, $\epsilon \ll 1$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $\frac{\pi}{2} \leqslant \gamma \leqslant \frac{\pi}{2} + \delta$, it holds that $\lambda_1(\gamma) > N - 1 - \epsilon$.*

Sketch of the proof. We apply the fact that $|\nabla u|^2 \geqslant |\partial_r u|^2 + |\partial_{\theta_1} u|^2/r^2$ holds for any smooth function u with compact support. Using polar coordinates and the change of variable $v := u/\cos(\pi\theta_1/2\gamma)$ we obtain the validity of (a) for any test function u. In order to prove (b) we apply (a) and estimates near $\gamma = \pi/2$. For more details, see [6], Section 3.2. □

Corollary 3.2. *As a consequence of* (b) *above, we get the continuous dependance of the Hardy constant* $\lim_{\gamma\to\pi/2}\mu(\mathcal{C}_\gamma) = \mu(\mathcal{C}_{\pi/2}) = N^2/4$.

Theorem 3.3. *Assume $\gamma \in (0,\pi)$. Then it holds*

$$\left(\frac{\sin\gamma}{\gamma}\right)^{N-2}\left(\frac{B_1}{\gamma}\right)^2 \leqslant \lambda_1(\gamma) \leqslant \left(\frac{\gamma}{\sin\gamma}\right)^{N-2}\left(\frac{B_1}{\gamma}\right)^2, \tag{8}$$

where B_1 is the first positive zero of the modified Bessel function $J_{(N-3)/2}$.

Sketch of the proof. We note that $\lambda_1(\gamma)$ coincides with the best constant in the 1-d Poincaré inequality on the interval $(0,\gamma)$ with respect to the measure $d\mu = \sin^{N-2} t\,dt$. Then, $\lambda_1(\gamma)$ turns out to be the first eigenvalue of a degenerate Sturm–Liouville problem at origin. When γ tends to zero, in terms of a suitable change of variables, we can transform the degenerate problem into a convenient Bessel equation (see [6], pp. 17–19). Then $\lambda_1(\gamma)$ may be determined asymptotically using formulas in [2], pp. 117–118. With this we finish the proof. □

Remark 5 *(Asymptotic behavior).* Note that the asymptotic formula $\lim_{\gamma\to 0}\lambda_1(\gamma)\gamma^2/B_1^2 = 1$ holds true, as a consequence of Theorem 3.3 above. We point out that, this formula has been also obtained (see [5] and the references therein) in a different context, when studying a problem on the biology of cell membranes.

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