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Complex Analysis/Theory of Signals

Wavelet frames with Laguerre functions ☆

Frames d'ondelettes et fonctions de Laguerre

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A R T I C L E   I N F O

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A B S T R A C T

Consider the functions Φ_n^α defined as $\mathcal{F}\Phi_n^\alpha(t) = t^{\frac{1}{2}} l_n^\alpha(2t)$, where l_n^α is a Laguerre function and $\Gamma(a,b) = \{(a^m bk, a^m)\}_{k,m\in\mathbb{Z}}$ is a hyperbolic lattice. We prove that, if the wavelet system $\mathcal{W}(\Phi_n^\alpha, \Gamma(a,b))$ is a frame of $H^2(\mathbb{C}^+)$, then $b \log a < 4\pi \frac{n+1}{\alpha+1}$.

R É S U M É

Soit le fonction Φ_n^α de la forme $\mathcal{F}\Phi_n^\alpha(t) = t^{\frac{1}{2}} l_n^\alpha(2t)$, ou l_n^α est une fonction de Laguerre et $\Gamma(a,b) = \{(a^m bk, a^m)\}_{k,m\in\mathbb{Z}}$ est une reseau hyperbolique. Notre resultat principal dit que, si l' ensemble d'ondelettes $\mathcal{W}(\Phi_n^\alpha, \Gamma(a,b))$ est un frame pour $H^2(\mathbb{C}^+)$, alors, $b \log a < 4\pi \frac{n+1}{\alpha+1}$.

1. Introduction

For every $x \in \mathbb{R}$ and $s \in \mathbb{R}^+$, let $z = x + is$ and define

$$\pi_z f(t) = s^{-\frac{1}{2}} f\big(s^{-1}(t-x)\big). \tag{1}$$

Fix a function $g \neq 0$. Then the continuous wavelet transform with respect to a wavelet g is defined as

$$W_g f(x,s) = \langle f, \pi_z g\rangle_{L^2(\mathbb{R})}, \tag{2}$$

where the function g is *admissible* (its Fourier transform belongs to $L^2(\mathbb{R}^+, \omega^{-1}\,d\omega)$). Let $\Gamma(a,b) = \{(a^m bk, a^m)\}_{k,m\in\mathbb{Z}}$. We say that $\mathcal{W}(g, \Gamma(a,b))$ is a *wavelet frame* for $H^2(\mathbb{C}^+)$, the standard Hardy space of the upper half-plane, if there exist constants $A, B > 0$ such that, for every $f \in H^2(\mathbb{C}^+)$,

$$A\|f\|^2_{H^2(\mathbb{C}^+)} \leqslant \sum_{z\in\Gamma(a,b)} \big|\langle f, \pi_z g\rangle\big|^2 \leqslant B\|f\|^2_{H^2(\mathbb{C}^+)}. \tag{3}$$

The problem of characterizing all lattices $\Gamma(a,b)$ for which $\mathcal{W}(g, \Gamma(a,b))$ is a frame is only solved for the special windows known as *Poisson wavelets*, defined via their Fourier transforms by

$$(\mathcal{F}\psi_\alpha)(t) = t^\alpha e^{-t}. \tag{4}$$

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In this case the results follow from the sampling results for the Bergman space of analytic functions [8,9]. It has been shown that only the Poisson wavelet leads to spaces of analytic functions [4].

In this note we investigate wavelet frames with analyzing wavelets Φ_n^α defined via their Fourier transforms $\mathcal{F}$ as

$$\mathcal{F}\Phi_n^\alpha(t) = t^{\frac{1}{2}} l_n^\alpha(2t), \quad \text{with } l_n^\alpha(t) = t^{\alpha/2} e^{-t/2} \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{t^k}{k!}. \tag{5}$$

The function $l_n^\alpha(t)$ is the Laguerre function (the Laguerre polynomial times the weight function) and it can also be evaluated from a three-term recurrence formula. The set $\{\Phi_n^\alpha\}_{n=0}^\infty$ forms a basis of space of admissible functions. These frames can be seen as a sort of "wavelet analogue" of the Gabor frames with Hermite functions [5], for which sufficient conditions are known. In the case of wavelet frames, we are interested in necessary conditions, since, unlike Gabor frames, there are no universal necessary conditions. Our main result is as follows:

Theorem 1.1. *If $\mathcal{W}(\Phi_n^\alpha, \Lambda)$ is a wavelet frame in $H^2(\mathbb{C}^+)$, then*

$$b \log a < 4\pi \frac{n+1}{\alpha+1}. \tag{6}$$

One should notice that using the density concepts of [9,7,10], we arrive at the constant $b \log a$ as the density of the hyperbolic lattice.

Our proof of Theorem 1 is based on an analogue of Proposition 3.2 in [5], which expresses the wavelet transform with respect to Φ_n^α as a combination of derivatives of an analytic function. To see that the inequality (6) is strict we use a stability property of wavelet frames, which is easily deduced from [4, Theorem 4.4] and allows to adapt the methods of [8]. The analogue of Proposition 3.2 in [5] is (see Section 2 for the definition of the Bergman transform):

Proposition 1.2. *Let $f \in H^2(\mathbb{C}^+)$ and $F = Ber_\alpha f$, the Bergman transform of f. Then*

$$W_{\Phi_n^\alpha} f(x,s) = \sum_{k=0}^{n} \frac{(2i)^k}{k!} \binom{n+\alpha}{n-k} s^{\frac{\alpha}{2}+k} F^{(k)}(z). \tag{7}$$

To ease the reading, we follow the presentation scheme of [5]. The next section contains the required tools and Section 3 sketches the proof of Theorem 1.1. In a forthcoming paper [3], we will investigate the corresponding wavelet superframe by constructing a polyanalytic Bergman transform which connects to a sampling problem in polyanalytic Bergman spaces of $\mathbb{C}^+$. The transform shares operator theoretical features with the polyanalytic Bargmann transform of [1] which connects to the Gabor superframe of [6]. Results using Seip's density [9] can be obtained for spaces in the unit disc $\mathbb{D}$ [2] which, unlike the case of analytic functions, are not equivalent to the upper half-plane.

2. Tools

The Hardy space $H^2(\mathbb{C}^+)$, is constituted by the analytic functions on the upper half-plane such that

$$\sup_{0<s<\infty} \int_{\mathbb{R}} \left| f(x+is) \right|^2 \mathrm{d}x < \infty. \tag{8}$$

Let $\alpha > -1$. The *weighted Bergman space* in the upper half-plane, $A_\alpha(\mathbb{C}^+)$, is constituted by analytic functions defined on the upper half-plane and such that

$$\int_{\mathbb{C}^+} \left| f(z) \right|^2 s^\alpha \, \mathrm{d}\mu^+(z) < \infty, \tag{9}$$

where $\mathrm{d}\mu^+(z)$ stands for area measure on $\mathbb{C}^+$. The *Bergman transform* of order α is the wavelet transform with a Poisson wavelet times a weight:

$$Ber_\alpha f(z) = s^{-\frac{\alpha}{2}} W_{\overline{\psi_{\frac{\alpha+1}{2}}}} f(-x,s) = \int_{\mathbb{R}^+} t^{\frac{\alpha+1}{2}} \mathcal{F}f(t) e^{izt} \, \mathrm{d}t. \tag{10}$$

It is an isomorphism $Ber_\alpha : H^2(\mathbb{C}^+) \to A_\alpha(\mathbb{C}^+)$ (this follows from the isometric properties of the wavelet transform and the fact that the image of Ber_α contains the reproducing kernel of $A_\alpha(\mathbb{C}^+)$).

The characteristic function of the hyperbolic lattice $\Gamma(a,b)$ is the analytic function in the upper half-plane defined by

$$h(z)=\left(\prod_{k=0}^{\infty}\frac{\sin\pi b^{-1}a^{-k}(ia^k-z)}{\sin\pi b^{-1}a^{-k}(ia^k+z)}\right)\left(\prod_{m=1}^{\infty}e^{\frac{2\pi}{b}}\frac{\sin\pi b^{-1}a^{m}(z-ia^{-m})}{\sin\pi b^{-1}a^{m}(z+ia^{-m})}\right). \tag{11}$$

It vanishes in $\Gamma(a,b)$ and satisfies [8] the estimate:

$$\left|h(z)\right|\lesssim s^{-\frac{2\pi}{b\ln a}}. \tag{12}$$

3. Proofs

3.1. Proof of Proposition 1.1

From a close inspection of formulas (5) and (4) one realizes that

$$l_n^{\alpha}(t)=t^{-\frac{1}{2}}\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n+\alpha}{n-k}(\mathcal{F}\psi_{\frac{\alpha}{2}+k+\frac{1}{2}})\left(\frac{t}{2}\right). \tag{13}$$

Thus, linearity and the inverse Fourier transform give:

$$\Phi_n^{\alpha}(t)=\sum_{k=0}^{n}\frac{(-2)^k}{k!}\binom{n+\alpha}{n-k}\psi_{\frac{\alpha}{2}+k+\frac{1}{2}}(t). \tag{14}$$

Combining (14) with (2) and (10) results in

$$W_{\Phi_n^{\alpha}}f(x,s)=\sum_{k=0}^{n}\frac{(-2)^k}{k!}\binom{n+\alpha}{n-k}s^{\frac{\alpha}{2}+k}Ber_{\alpha+2k}f. \tag{15}$$

Now, if $f\in H^2(\mathbb{C}^+)$ and $F=Ber_{\alpha}f$, differentiation of (10) under the integral transform shows that $F^{(k)}(z)=i^k Ber_{\alpha+2k}\ f(z)$. This yields (7).

3.2. Stability of wavelet frames

We call *wavelet space*, $\mathcal{W}_g$, to the image space of the wavelet transform associated with g. The sequence $\Gamma=\{(x,s)\}$ is a sampling set for $\mathcal{W}_g$ if there exist constants A, B such that, for every $f\in H^2(\mathbb{C}^+)$,

$$A\|W_g f\|^2_{L^2(\mathbb{C}^+,s^{-1})}\leqslant\sum_{(x,s)\in\Gamma}\left|W_g f(x,s)\right|^2\leqslant B\|W_g f\|^2_{L^2(\mathbb{C}^+,s^{-1})}. \tag{16}$$

Since $\|W_g f\|_{L^2(\mathbb{C}^+,s^{-1})}$ and $\|f\|_{H^2(\mathbb{C}^+)}$ are equivalent norms, then the definition of the wavelet transform (2) shows that sampling sets correspond to wavelet frames. Thus, we can recast the stability result with respect to the jittered error of sampling sequences for wavelet spaces [4, Theorem 4.4], as:

Proposition 3.1. *Suppose that $W(g,\Lambda)$ is a wavelet frame. Then there exists $\delta>0$ such that if the pseudohyperbolic distance $\rho(z,w)$ is $<\delta$ for all $z\in\Lambda$, $w\in\Gamma$, then $W(g,\Gamma)$ is also a frame.*

3.3. Proof of Theorem 1.1

Argue by contradiction assuming that $b\log a>4\pi\frac{n+1}{\alpha+1}$. This implies the existence of $\epsilon>0$ such that $\frac{2\pi(n+1)}{b\log a}=\frac{\alpha+1-\epsilon}{2}$. Now apply (12) to $H(z)=(z+i)^{\epsilon}h^{n+1}(z)$, yielding $|H(z)|\lesssim|z+i|^{\epsilon}s^{-\frac{\alpha+1-\epsilon}{2}}$. Changing variables to the unit disc by setting $z=i\frac{w+1}{1-w}$, and applying the bound on H, one obtains the following estimate:

$$\int_{\mathbb{C}^+}\left|H(z)\right|^2 s^{\alpha}\,\mathrm{d}\mu^+(z)\lesssim\int_{\mathbb{D}}\left(1-|w|^2\right)^{-1+\epsilon}\mathrm{d}A(w)<\infty,$$

where $\mathrm{d}A(w)$ stands for area measure in the unit disc. Thus, $H(z)\in A_{\alpha}(\mathbb{C}^+)$. Now, since $Ber_{\alpha}:H^2(\mathbb{C}^+)\to A_{\alpha}(\mathbb{C}^+)$ is an isomorphism, there exists $f\in H^2(\mathbb{C}^+)$ such that $Ber_{\alpha}f=H(z)$. Using (7) of Proposition 1.1 one easily sees that $W_{\Phi_n^{\alpha}}f$ vanishes in $\Gamma(a,b)$. Consequently, $\mathcal{W}(\Phi_n^{\alpha},\Gamma(a,b))$ cannot be a wavelet frame for $H^2(\mathbb{C}^+)$. Thus, $b\log a\leqslant4\pi\frac{n+1}{\alpha+1}$.

To prove that the inequality is strict, observe that, by Proposition 3.1, there exists a $\delta>0$ such that, if $\mathcal{W}(\Phi_n^{\alpha},\Gamma(a,b))$ is a frame and $\Gamma_0=\{w_{mk}\}$ satisfies $\rho(a^m(bk+i),w_{mk})<\delta$ for all m,k, where $\rho(.,.)$ is the pseudohyperbolic distance in the upper half-plane, then $\mathcal{W}(\Phi_n,\Gamma_0)$ is also a frame. Thus, if $b\log a=4\pi\frac{n+1}{\alpha+1}$, we can choose δ_0 such that $w_{mk}=a^m(bk+i(1-\delta_0))$ satisfies $\rho(\Gamma(a,b),w_{mk})<\delta$ and therefore is a sampling sequence. This is impossible by the argument in the previous paragraph, since $\{w_{mk}\}=\Gamma(a,b/(1-\delta_0))$ and $b/(1-\delta_0)\log a>b\log a=4\pi\frac{n+1}{\alpha+1}$.

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