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Analysis of some injection bounds for Sobolev spaces by wavelet decomposition

Analyse de quelques bornes d'injection des espaces de Sobolev, en utilisant la décomposition par ondelettes

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A R T I C L E   I N F O

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A B S T R A C T

We consider the Sobolev spaces $H^s(\Omega)$ and $H^s_0(\Omega)$ and the Besov spaces $B^{1/2}_{2,\infty}(\Omega)$, where Ω is a sufficiently regular (see Lemma 2) subdomain of $\mathbb{R}^2$. It is well known that for the values of $s \in [0, 1/2)$ the two Sobolev spaces coincide, with equivalence of the norms, and that the inclusion $B^{1/2}_{2,\infty}(\Omega) \subset H^s(\Omega)$ holds. The Note is concerned with the explicit analysis of the constants appearing in the continuity bounds for the injections $H^s(\Omega) \hookrightarrow H^s_0(\Omega)$ and $B^{1/2}_{2,\infty}(\Omega) \hookrightarrow H^s(\Omega)$ and of their dependence on the regularity s of the spaces. The analysis is carried out by using the wavelet characterization of the corresponding norms.

R É S U M É

On considère les espaces de Sobolev $H^s(\Omega)$ et $H^s_0(\Omega)$, et l'espace de Besov $B^{1/2}_{2,\infty}(\Omega)$, ou Ω est un domaine suffisamment régulier (voir Lemme 2) de $\mathbb{R}^2$. On sait que pour des valeurs de $s \in [0, 1/2)$ les deux espaces de Sobolev coïncident, avec équivalence des normes, et qu'on a l'inclusion $B^{1/2}_{2,\infty}(\Omega) \subset H^s(\Omega)$. Cet article donne une analyse explicite des constantes qui apparaissent dans les bornes d'inclusion $H^s(\Omega) \hookrightarrow H^s_0(\Omega)$ and $B^{1/2}_{2,\infty}(\Omega) \hookrightarrow H^s(\Omega)$ et, plus précisément, de leur dépendance du paramètre de régularité s. On utilise pour cela la caractérisation par ondelettes des normes correspondantes.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. For $s \in [0, 1/2)$, we consider the Sobolev spaces $H^s(\Omega)$ and $H^s_0(\Omega)$, defined respectively, by space interpolation [7], as $H^s(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1-s}$ and $H^s_0(\Omega) = [H^1_0(\Omega), L^2(\Omega)]_{1-s}$. It is well known [6] that $H^s(\Omega)$ and $H^s_0(\Omega)$ coincide, and that the corresponding norms are equivalent, that is that there exist two constants c_s and C_s, depending on the regularity parameter s and on the domain Ω, such that

$$c_s \|u\|_{H^s(\Omega)} \leqslant \|u\|_{H^s_0(\Omega)} \leqslant C_s \|u\|_{H^s(\Omega)}. \tag{1}$$

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The injection $B^{1/2}_{2,\infty}(\Omega) \hookrightarrow H^s(\Omega)$ holds as well [7], that is there exists a constant B_s, depending on s such that

$$\|u\|_{H^s(\Omega)} \leqslant B_s \|u\|_{B^{1/2}_{2,\infty}(\Omega)}. \tag{2}$$

The way the constants C_s and B_s behave as s converges to $1/2$ plays a key role in several applications [1,2,5]. We are interested here in explicitly studying such a dependence by means of wavelet analysis. In what follows the notation $A \lesssim B$ ($A \gtrsim B$) signifies that the quantity A is bounded from above (below) by $C \cdot B$, where C is a constant that does not depend on the regularity parameter s. $A \simeq B$ stands for $B \lesssim A \lesssim B$.

1. Wavelet characterization of Sobolev and Besov norms

It is well known that it is possible to express equivalent norms for Sobolev and Besov spaces in terms of suitable norms on the sequences of wavelet coefficients. More precisely we can select a sufficiently regular Daubechies orthonormal wavelet basis for $L^2(\mathbb{R})$, starting from which it is possible [4] to construct two orthonormal bases $\mathcal{B} = \{\varphi_{\mathbf{k}}, \mathbf{k} \in \mathbb{K}_{j_0}\} \cup \{\psi_{j,\mathbf{k}}, j \geqslant j_0, \mathbf{k} \in \mathbb{I}_j\}$, and $\mathcal{B}^0 = \{\varphi^0_{\mathbf{k}}, \mathbf{k} \in \mathbb{K}_{j_0}\} \cup \{\psi^0_{j,\mathbf{k}}, j \geqslant j_0, \mathbf{k} \in \mathbb{I}_j\}$, (the elements of $\mathcal{B}^0$ satisfying homogeneous boundary conditions) such that every $u \in L^2([0,1]^2)$ can be written in both ways as

$$\begin{aligned} u &= \sum_{\mathbf{k}\in\mathbb{K}_{j_0}} (u, \varphi_{\mathbf{k}})\varphi_{\mathbf{k}} + \sum_{j\geqslant j_0}\sum_{\mathbf{k}\in\mathbb{I}_j} (u, \psi_{j,\mathbf{k}})\psi_{j,\mathbf{k}} \\ &= \sum_{\mathbf{k}\in\mathbb{K}_{j_0}} \big(u, \varphi^0_{\mathbf{k}}\big)\varphi^0_{\mathbf{k}} + \sum_{j\geqslant j_0}\sum_{\mathbf{k}\in\mathbb{I}_j} \big(u, \psi^0_{j,\mathbf{k}}\big)\psi^0_{j,\mathbf{k}}, \end{aligned}$$

and the following norm equivalences hold for all $s \in [0,1]$:

$$\|u\|^2_{H^s([0,1]^2)} \simeq \sum_{\mathbf{k}\in\mathbb{K}_{j_0}} \big|(u, \varphi_{\mathbf{k}})\big|^2 + \sum_{j\geqslant j_0}\sum_{\mathbf{k}\in\mathbb{I}_j} 2^{2js}\big|(u, \psi_{j,\mathbf{k}})\big|^2, \tag{3}$$

$$\|u\|^2_{H^s_0([0,1]^2)} \simeq \sum_{\mathbf{k}\in\mathbb{K}_{j_0}} \big|\big(u, \varphi^0_{\mathbf{k}}\big)\big|^2 + \sum_{j\geqslant j_0}\sum_{\mathbf{k}\in\mathbb{I}_j} 2^{2js}\big|\big(u, \psi^0_{j,\mathbf{k}}\big)\big|^2, \quad s \neq \frac{1}{2}. \tag{4}$$

Here $\mathbb{K}_{j_0} = \{1, \ldots, 2^{j_0}\}^2$. $\mathbb{I}_j \cong \{1, \ldots, 2^{j+1}\}^2 \setminus \{1, \ldots, 2^j\}^2$ denotes the multi-index set corresponding to the wavelet functions at level j, which are obtained from the corresponding one dimensional basis by tensor product.

By using the Littlewood–Paley decomposition in terms of the wavelet basis, one can also derive the characterization of Besov spaces in terms of wavelets coefficients. Precisely,

$$\|u\|^2_{B^{1/2}_{2,\infty}([0,1]^2)} \simeq \sum_{\mathbf{k}\in\mathbb{K}_{j_0}} \big|(u, \varphi_{\mathbf{k}})\big|^2 + \sup_{j\geqslant j_0} 2^j \sum_{\mathbf{k}\in\mathbb{I}_j} \big|(u, \psi_{j,\mathbf{k}})\big|^2. \tag{5}$$

2. The dependence of the norm equivalence constants C_s and B_s on s

By using (3), (4) and (5) it is possible to prove the following results [3]:

Lemma 1. *If $u \in H^s([0,1]^2)$, $0 < s < 1/2$, then*

$$\|u\|_{H^s_0([0,1]^2)} \lesssim \frac{1}{1/2 - s}\|u\|_{H^s([0,1]^2)}. \tag{6}$$

If $u \in B^{1/2}_{2,\infty}([0,1]^2)$ then for all s, $0 < s < 1/2$,

$$\|u\|_{H^s([0,1]^2)} \lesssim \frac{1}{\sqrt{1/2 - s}}\|u\|_{B^{1/2}_{2,\infty}([0,1]^2)}. \tag{7}$$

The proof of (6) consists in expressing u as a linear combination of the functions of the basis $\mathcal{B}$ and injecting such an expression in the norm equivalence (4). After observing that the two bases have in common all basis functions whose support is strictly included in $[0,1]^2$, a suitable a priori bound on the terms $|(\psi_{j,\mathbf{k}}, \psi^0_{\ell,\mathbf{n}})|$ of the form

$$\big|\big(\psi_{j,\mathbf{k}}, \psi^0_{\ell,\mathbf{n}}\big)\big| \lesssim 2^{-|j-\ell|},$$

and on all other scalar products between the basis functions of $\mathcal{B}$ and $\mathcal{B}^0$ allows to prove the bound (6) by applying a Schur Lemma type argument. The term $(1/2 - s)^{-1}$ appears by taking the limit of the sum of the series $\sum_{j\geqslant j_0} 2^{-(1-2s)j}$. The bound (7) is proven by a similar argument.

The result of Lemma 1 can be extended to fairly regular generic domain Ω, thanks to the following lemma [3]:

Lemma 2. *Let $T : \Omega \to [0,1]^2$ be a bounded and boundedly invertible map, with bounded Jacobian such that the equivalence $|(x_1, y_1) - (x_2, y_2)| \simeq |T(x_1, y_1) - T(x_2, y_2)|$ holds for all $(x_1, y_1), (x_2, y_2) \in \Omega$ (T is bi-Lipschitz). Let $u \in L^2([0,1]^2)$ and let $\hat{u}(x) := u \circ T(x)$. Then, for each $s \in [0, 1/2)$,*

$$\hat{u} \in H^s(\Omega) \quad \Leftrightarrow \quad u \in H^s\big([0,1]^2\big), \qquad \|\hat{u}\|_{H^s(\Omega)} \simeq \|u\|_{H^s([0,1]^2)},$$
$$\hat{u} \in H^s_0(\Omega) \quad \Leftrightarrow \quad u \in H^s_0\big([0,1]^2\big), \qquad \|\hat{u}\|_{H^s_0(\Omega)} \simeq \|u\|_{H^s_0([0,1]^2)},$$
$$\hat{u} \in B^{1/2}_{2,\infty}(\Omega) \quad \Leftrightarrow \quad u \in B^{1/2}_{2,\infty}\big([0,1]^2\big), \qquad \|\hat{u}\|_{B^{1/2}_{2,\infty}(\Omega)} \simeq \|u\|_{B^{1/2}_{2,\infty}([0,1]^2)}.$$

Corollary 3. *For every domain $\Omega \subset \mathbb{R}^2$ such that there exists a map $T : \Omega \to [0,1]^2$ satisfying the assumptions of Lemma 2, it holds that if $u \in H^s(\Omega)$, $0 < s < 1/2$, then*

$$\|u\|_{H^s_0(\Omega)} \lesssim \frac{1}{1/2 - s} \|u\|_{H^s(\Omega)}, \tag{8}$$

and if $u \in B^{1/2}_{2,\infty}(\Omega)$ then for all s, $0 < s < 1/2$,

$$\|u\|_{H^s(\Omega)} \lesssim \frac{1}{\sqrt{1/2 - s}} \|u\|_{B^{1/2}_{2,\infty}(\Omega)}. \tag{9}$$

3. The optimality of the injection bounds

The dependence of the constants appearing in (8) on the regularity parameter $s \in [0, 1/2)$ is sharp and cannot be improved. In order to prove this, it is sufficient to exhibit a function u whose H^s-norm is bounded uniformly with respect to s, while its H^s_0-norm behaves like $\frac{1}{1/2-s}$. The existence of such a function [3] can be proven constructively, also with the aid of wavelets, as stated by the following proposition:

Proposition 4. *There exists a function $u \in H^{1/2}(\Omega)$, $u \notin H^{1/2}_{00}(\Omega)$ such that*

$$\|u\|_{H^s_0(\Omega)} \gtrsim \frac{1}{1/2 - s}.$$

Analogously, the optimality of the bound (9) holds as well. In particular the following result holds:

Proposition 5. *There exists a function $u \in B^{1/2}_{2,\infty}(\Omega)$, $u \notin H^{1/2}(\Omega)$ such that*

$$\|u\|_{H^s(\Omega)} \gtrsim \frac{1}{\sqrt{1/2 - s}}.$$

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