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Differential Geometry

Ricci flow of non-collapsed 3-manifolds: Two applications

Flot de Ricci de variétés de dimension 3 non-effondrées : Deux applications

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ABSTRACT

In this short Note, we give two simple applications of results of Miles Simon about the Ricci flow of non-collapsed 3-manifolds. First, we prove a new diffeomorphism finiteness result for 3-manifolds with Ricci curvature bounded from below, volume bounded from below and diameter bounded from above. Second, we give an alternate proof of a theorem of Cheeger and Colding. Namely, we prove that if a sequence M_i of compact 3-manifolds with Ricci curvature bounded from below Gromov–Hausdorff converges to a compact 3-manifold M, then all the M_i's are diffeomorphic to M for i large enough.

RÉSUMÉ

Dans cette Note, on donne deux applications simples de résultats dûs à Miles Simon sur le flot de Ricci des variétés de dimension 3 non-effondrées. On montre d'abord un nouveau théorème de finitude à difféomorphisme près pour les variétés de dimension 3 à courbure de Ricci minorée, diamètre majoré et volume minoré. Ensuite, on donne une nouvelle preuve d'un résultat dû à Cheeger et Colding. Si une suite de variétés compactes de dimension 3 à courbure de Ricci minorée converge au sens de Gromov–Hausdorff vers une une variété compacte de dimension 3, alors tout les éléments de la suite sont difféomorphes à la variété limite à partir d'un certain rang.

In the recent years, Ricci flow has proved to be a valuable tool in the study of the geometry of Riemannian 3-manifolds. Starting with Hamilton's foundational work in 1983 [8], it has lead to Perelman's proof of Thurston's geometrization conjecture in 2003 (see [10] and subsequent papers).

In this short Note, we give two applications of the Ricci flow in dimension 3 using results from [11]. In the first section, we give the main result from [11] that we will need in the proofs. In the second section, we briefly discuss previously known finiteness theorems and give a proof of a new finiteness result in dimension 3 using Ricci flow. In the third section, we use these tools to give an alternate proof of a theorem of Cheeger and Colding in dimension 3.

These results follow essentially from Proposition 9.3(ii) of [11], but more details are included here for the reader's convenience.

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1. Results from the Miles Simon's paper [11]

For convenience of the reader, we recall here one of the main theorems of [11] that will be used in the sequel:

Theorem 1. *(See [11], Theorem 1.9.) For any $k > 0$ and $v_0 > 0$, there exist $T > 0$ and $K > 0$ such that, if (M, g_0) is a complete 3-manifold with bounded curvature satisfying*:

(i) $\mathrm{Ricci}(g_0) \geqslant -k$
(ii) $\forall x \in M,\ \mathrm{vol}_{g_0}(B(x,1)) \geqslant v_0$

then the solution $(M, g(t))$ to the Ricci flow with initial condition g_0 exists at least on $[0, T)$ and satisfies, for all t in $(0, T)$:

(a) $\mathrm{Ricci}(g(t)) \geqslant -K$
(b) $\sup_M \|\mathrm{Rm}(g(t))\| \leqslant \frac{K}{t}$
(c) $\forall x \in M,\ \mathrm{vol}_{g(t)}(B(x,1)) \geqslant \frac{v_0}{2}$
(d) *if* $0 \leqslant s < t < T$, $e^{K(t-s)} d_{g(s)} \geqslant d_{g(t)} \geqslant d_{g(s)} - K(\sqrt{t} - \sqrt{s})$

In [11], this result has been used to prove that a Gromov–Hausdorff limit of 3-manifolds satisfying (i) and (ii) is itself a smooth 3-manifold. The strength of this theorem is that the estimates depend on the geometry of (M, g_0) in a very weak way.

2. A finiteness theorem in dimension 3

The first finiteness theorems in Riemannian geometry were independently obtained by Weinstein [12] and Cheeger [4] in the late 1960's. Cheeger's result was that given a two sided bound on the sectional curvatures, an upper bound on the diameter and a lower bound on the volume, only finitely many diffeomorphism types of manifold admit a Riemannian metric satisfying these bounds. There has been a great number of successful attempts to relax the assumptions of this theorem, for an overview we refer to [3]. One way is to try to replace bounds on the sectional curvature by bounds on the Ricci curvature.

Theorem 1 can be used to show:

Theorem 2. *Given $V > 0$, $D > 0$ and $k \in \mathbb{R}$ there exists only finitely many closed 3-manifolds which admit Riemannian metrics such that* $\mathrm{vol} \geqslant V$, $\mathrm{diam} \leqslant D$ *and* $\mathrm{Ricci} \geqslant k$ *up to diffeomorphism.*

Results in this direction have been obtained by Anderson and Cheeger [1,2] in the beginning of the 1990's. These results are true in any dimension but require stronger assumptions. In [1], in addition to the assumptions of Theorem 2 upper bounds on the Ricci curvature and the $L^{d/2}$ norm of the curvature operator are required. Ref. [2] assumes lower bound on injectivity radius instead of volume.

Proof. The proof goes by contradiction. Assume that we can find an infinite sequence (M_i, g_i) of manifolds satisfying $\mathrm{Ricci}(g_i) \geqslant k$, $\mathrm{vol}(M_i, g_i) \geqslant V$, $\mathrm{diam}(M_i, g_i) \leqslant D$ and such that any two of the M_i's are not diffeomorphic. Choosing some $r > D$, we have that for each i, the volume of $B_{g_i}(x, r)$ is greater than V. This shows our sequence uniformly satisfies the hypothesis of Theorem 1.

We now apply Theorem 1 to each manifold of the sequence and get a sequence of Ricci flows $(M_i, g_i(t))_{t \in [0,T)}$ satisfying estimates (a), (b), (c) and (d).

The estimate (b) in the theorem gives a two sided curvature bound which is uniform on any compact interval of $(0, T)$. Moreover, using a theorem of Cheeger, Gromov and Taylor (see [6], p. 199) and estimates (b) and (c), one gets, at time $t_0 = T/2$, a uniform lower bound on the injectivity radius. Thus, up to a subsequence, the sequence of Ricci flows $(M_i, g_i(t))_{t \in (0,T)}$ smoothly converges to a Ricci flow $(\widetilde{M}, \tilde{g}(t))_{t \in (0,T)}$ thanks to Hamilton's compactness theorem [9].

Furthermore, using estimate (d), if we pick any $t \in (0, T)$, we have:

$$e^{Kt} \operatorname{diam}(M_i, g_i) \geqslant \operatorname{diam}\bigl(M_i, g_i(t)\bigr)$$

In particular, for any i we get: $\mathrm{diam}(M_i, g_i(t)) \leqslant e^{Kt} D$. This implies, as $(M_i, g_i(t))$ converges smoothly to $(\widetilde{M}, \tilde{g}(t))$ up to a subsequence, $\mathrm{diam}(\widetilde{M}, \tilde{g}(t)) \leqslant e^{Kt} D$. This shows that the limit manifold $\widetilde{M}$ is compact and that, up to a subsequence, the M_i's are all diffeomorphic to $\widetilde{M}$ for i large enough. This is a contradiction. □

Remark 1. There is a shorter way for this proof: the estimates we get with Theorem 1 can be used to say that a manifold satisfying the assumptions of Theorem 2 bears a metric satisfying the assumptions of Cheeger finiteness theorem (just by flowing the metric for some fixed time $t_0 \in (0, T)$). However, we will need the convergence of Ricci flows in the next section.

3. A Ricci flow proof of a theorem of Cheeger and Colding

In [5], Cheeger and Colding proved the following theorem:

Theorem 3. *Let $(M_i^n, g_i)_{i\in\mathbb{N}}$ be a sequence of compact n-dimensional Riemannian manifolds with Ricci curvature bounded from below which converges to a compact Riemannian n-manifold (M^n, g) in the Gromov–Hausdorff sense.*[1] *Then, for i large enough, all the manifolds M_i are diffeomorphic to M.*

In this section, we give a Ricci flow proof of this theorem in the case $n = 3$.

Proof. Let (M_i^3, g_i) be a sequence of 3-manifolds whose Ricci curvature is bounded from below and which GH-converges to (M^3, g) a smooth Riemannian 3-manifold. Since $(M_i, g_i)_{i\in\mathbb{N}}$ GH-converges to (M, g), $\operatorname{diam}(M_i, g_i)$ tends to $\operatorname{diam}(M, g)$ as i goes to infinity (by definition of GH-convergence) and $\operatorname{vol}(M_i, g_i)$ tends to $\operatorname{vol}(M, g)$ (this is a theorem of Colding, see [7]). Therefore, there exist positive constants D and V such that, for any i, $\operatorname{vol}(M_i, g_i) > V$ and $\operatorname{diam}(M_i, g_i) < D$.

We argue by contradiction. If the theorem is false, we can find a subsequence of (M_i, g_i) such that none of the M_i's is diffeomorphic to M.

Then, as in the proof of Theorem 2, the Ricci flow $(M_i, g_i(t))$ of each manifold of the sequence starting at g_i exists on $(0, T)$ and satisfies the estimates (a), (b), (c) and (d). We then have a subsequence of $(M_i, g_i(t))_{t\in(0,T)}$ which smoothly converges to a Ricci flow $(\widetilde{M}, \tilde{g}(t))_{t\in(0,T)}$ satisfying the same estimates. Furthermore, $\widetilde{M}$ is compact and the M_i's are diffeomorphic to $\widetilde{M}$ when i is large enough.

All that remains to be done is to show that $\widetilde{M}$ is diffeomorphic to M. This is done in the proof of Theorem 9.2 of [11]. In a few words, one shows, using estimate (d), that the distances $d_{\tilde{g}(t)}$ uniformly converge as t goes to 0 to a distance l on $\widetilde{M}$ which defines the same topology as the $d_{\tilde{g}(t)}$'s. Then, (M, g) is obtained as the limit of the $(M_i, g_i(t))$ when first t goes to 0 and then i goes to infinity and $(\widetilde{M}, l)$ is obtained as the limit of the $(M_i, g_i(t))$ when first i goes to infinity and then t goes to 0. Using estimate (d) one can then show that $(\widetilde{M}, l)$ is isometric to (M, g) which finally gives that M is homeomorphic to $\widetilde{M}$ (see [11], proof of Theorem 8.2). Since in dimension 3 every manifold has a unique smooth structure, $\widetilde{M}$ is diffeomorphic to M. □

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[1] We will write this as "GH-converges".