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Number Theory

Bounded p-adic L-functions of motives at supersingular primes

Fonctions L p-adiques bornées des motifs en une place très supersingulière

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A B S T R A C T

Pollack (2003) [17] proved that the p-adic L-function attached to a modular form $f = \sum a_n q^n$ at the most supersingular prime p (i.e. $a_p = 0$) is controlled by two Iwasawa functions and by two half-logarithms. We formulate a (conjectural) generalization of this result to p-adic L-functions attached to motives, and give examples confirming our expectation (symmetric powers and tensor products of modular forms).

R É S U M É

Dans [17] Pollack (2003) a montré que la fonction L p-adique associée à une forme modulaire $f = \sum a_n q^n$ en une place très supersingulière p ($a_p = 0$) est contrôlée par deux fonctions d'Iwasawa et deux semi-logarithmes. Nous énonçons une généralisation conjecturale des résultats de Pollack aux fonctions L p-adiques des motifs. Nous donnons divers exemples (produits symétriques et produits tensoriels de formes modulaires) qui confirment cette conjecture.

1. Introduction

Let f be a modular form of weight k, level N, and character ϵ which is an eigenform for each Hecke operator T_n with eigenvalue a_n. Fix a prime p, $(p, N) = 1$. Let α_p, α'_p be the inverse roots of the local p-polynomial $1 - a_p x + \epsilon(p) p^{k-1} x^2$; assume that $\mathrm{ord}_p\, \alpha_p \leqslant \mathrm{ord}_p\, \alpha'_p$. Put $h = \mathrm{ord}_p\, \alpha_p$. Let $L_p(f, \cdot)$ be the corresponding p-adic L-function (see [1,18,11]); it is a $\mathbb{C}_p$-analytic function defined on the p-adic Lie group $X_p := \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$, in general unbounded (but h-admissible in the sense of Amice and Vélu [1] and Vishik [18]). Here we mean that a $\mathbb{C}_p$-analytic function is first defined on $\{z \in \mathbb{C}_p^\times\colon |z-1|_p < 1\}$ as the sum of a convergent power series, and extended to the whole group X_p by shifts.

$L_p(f, \chi)$ is analytic in χ, and hence we can form its power series expansion about a tame character ψ; we denote this power series by $L_p(f, \psi, T)$. For $T = u - 1$, we have $L_p(f, \psi, u-1) = L_p(f, \psi\chi_u)$, where χ_u denotes a wild part of χ.

Consider the most supersingular case $a_p = 0$. Then $\alpha_p = -\alpha'_p$, and hence $\mathrm{ord}_p\, \alpha_p = \mathrm{ord}_p\, \alpha'_p = \frac{k-1}{2}$. Pollack ([17], Theorem 5.1) established the following decomposition result: $L_p(f, \psi, T) = L_p^+(f, \psi, T) \cdot \log_p^+(T) + L_p^-(f, \psi, T) \cdot \log_p^-(T) \cdot \alpha_p$, where $L_p^\pm(f, \psi, T)$ are bounded, and $\log_p^+(T) \sim \log_p^-(T) \sim \log_p(1+T)^{(k-1)/2}$.

In this Note we formulate a conjectural generalization of his result to p-adic L-functions attached to pure critical motives at good, very supersingular primes, and give examples confirming our expectation (symmetric powers and tensor products

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of modular forms). We hope it will provide a useful framework for further research on p-adic L-functions and generalized Main Conjectures in the non-ordinary case.

2. A conjecture on p-adic L-functions of motives

Let M be a pure motive over $\mathbb{Q}$ (with coefficients in $\mathbb{Q}$, for simplicity) of weight $w = w(M)$ and rank $d = d(M)$, given by Betti, de Rham and l-adic realizations (for each prime l) $H_B(M)$, $H_{DR}(M)$ and $H_l(M)$ which are, respectively, vector spaces over $\mathbb{Q}$, $\mathbb{Q}$ and $\mathbb{Q}_l$ of dimension d, and which are endowed with the additional structures and comparison isomorphisms (for details see [8,4,3]). In particular $H_B(M)$ admits an involution ρ_B, $H_l(M)$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, and there is a Hodge decomposition into $\mathbb{C}$-vector spaces $H_B(M)\otimes\mathbb{C} = \bigoplus_{i+j=w} H^{i,j}(M)$, where, letting ρ_B act on the vector space on the left via the first factor in the tensor product, we have $\rho_B(H^{i,j}(M)) = H^{j,i}(M)$. Let $h(i,j) = \dim H^{i,j}(M)$, and let $d^{\pm} = d^{\pm}(M)$ be the $\mathbb{Q}$-dimension of the $\pm$-subspace of ρ_B.

The L-function of M is defined for $\mathrm{Re}(s) \gg 0$ as the Euler product $L(M,s) = \prod_p L_p(M, p^{-s})$, extended over all primes p, and where the local p-polynomial $L_p(M,X)^{-1} := \det(1 - \rho_l(\mathrm{Fr}_p^{-1})X \mid H_l(M)^{I_p}) = \sum_{i=0}^{d} A_i(p)X^i$; here ρ_l is the representation giving $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module structure on $H_l(M)$, and $\mathrm{Fr}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the Frobenius element at p. Of course, the degree of the Euler factor at p is d only at good primes (outside the ramification set of the motive, with $l \neq p$). We put $\Lambda(M,s) = L_\infty(M,s)L(M,s)$, where $L_\infty(M,s)$ denotes the factor at infinity.

Let us fix a sign $\epsilon_0 = \pm$. Assume that the twisted motive $M(\chi)(m)$ is critical (i.e. that 0 is a critical point for $M(\chi)(m)$ in the sense of Deligne [8]) for some Dirichlet character χ and an integer m satisfying $\epsilon_0 = \mathrm{sign}((-1)^m\epsilon(\chi))$. Deligne's period conjecture (see [8]) asserts that the quantity $\frac{\Lambda(M(\chi),m)}{G(\chi)^{d^{\epsilon_0}}\Omega(\epsilon_0,M)}$ is algebraic, where $G(\chi)$ is the Gauss sum, and $\Omega(\epsilon_0, M)$ denotes one of the modified periods of M (see [8,3] for a more precise statement).

Let $P_{N,p}(u,M)$ denote the *p-Newton polynomial* of M: it is the convex hull of the points $(i, \mathrm{ord}_p A_i(p))$, $0 \leqslant i \leqslant d$. It is well known, that the length of the horizontal segment of slope k is equal to the number of the inverse roots $\alpha_p^{(j)}$ such that $\mathrm{ord}_p\,\alpha_p^{(j)} = k$. The *Hodge polygon* $P_H(u,M)$ by definition passes through the points $(0,0),\ldots,(\sum_{i'\leqslant i} h(i',j), \sum_{i'\leqslant j} i'h(i',j)),\ldots$, so that the length of the horizontal segment of slope i equals $h(i,j)$.

Now we formulate a general conjecture on the existence of (unbounded, in general) p-adic L-functions attached to pure critical motives over $\mathbb{Q}$. For p good for M, we assume, that the inverse roots of $L_p(M,X)^{-1}$ are indexed in such a way that $\mathrm{ord}_p\,\alpha_p^{(1)} \leqslant \mathrm{ord}_p\,\alpha_p^{(2)} \leqslant \cdots \leqslant \mathrm{ord}_p\,\alpha_p^{(d)}$. For any Dirichlet character χ and an integer m, we define the p-factor

$$A_p\big(M(\chi),m\big) = \begin{cases} \prod_{i=d^+ +1}^{d}(1-\chi(p)\alpha_p^{(i)}p^{-m})\prod_{i=1}^{d^+}(1-\chi^{-1}(p)\alpha_p^{(i)^{-1}}p^{m-1}) & \text{if } p \nmid c(\chi), \\ \prod_{i=1}^{d^+}(\frac{p^m}{\alpha_p^{(i)}})^{\mathrm{ord}_p c(\chi)} & \text{if } p \mid c(\chi). \end{cases}$$

We use the following invariant (generalized Hasse invariant of M, introduced by the author in 1991; see [7] or [13], p. 266): $h_p(M) := P_{N,p}(d^+,M) - P_H(d^+,M)$. It is known (Katz–Mazur) that $P_{N,p}(u,M) \geqslant P_H(u,M)$.

Let us fix a sign $\epsilon_0 = \pm$. Let $[m_\star, m^\star]$ be the critical strip for M, where $m_\star = \max\{j\colon \exists j,k,\ j<k \text{ such that } h(j,k)\neq 0\}+1$, and $m^\star = \min\{j\colon \exists j,k,\ j>k \text{ such that } h(j,k)\neq 0\}$. We fix embeddings $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\overline{\mathbb{Q}} \to \mathbb{C}_p$. Let $x_p : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ denote the inclusion.

Conjecture 1. *(See [7,13].) There exists a $\mathbb{C}_p$-meromorphic function $L_p^{(\epsilon_0)} : X_p \to \mathbb{C}_p$ such that*

(i) *for all but a finite number of pairs $(m,\chi) \in \mathbb{Z}\times X_p^{tors}$ such that $M(\chi)(m)$ is critical and $\epsilon_0 = \mathrm{sgn}((-1)^m\epsilon(\chi))$, we have*

$$L_p^{(\epsilon_0)}\big(\chi x_p^m\big) = G(\chi)^{-d^{\epsilon_0}(M)}A_p\big(M(\chi),m\big)\frac{\Lambda(M(\chi),m)}{\Omega(\epsilon_0,M)};$$

(ii) *if $h(w/2,w/2) = 0$, then $L_p^{(\epsilon_0)}$ is holomorphic*; *otherwise the function $\prod_\xi (x(g_0)-\xi(g_0))^{n(\xi)}L_p^{(\epsilon_0)}(x)$ is holomorphic, where ξ runs over finite set of p-adic characters, $n(\xi)$ are positive integers, and $g_0 \in \mathbb{Z}_p^\times$*;

(iii) *if $P_{N,p}(d^+,M) = P_H(d^+,M)$, then the holomorphic function in* (ii) *is bounded*;

(iv) *the function from* (ii) *is holomorphic of the type $O(\log_p^{h_p(M)})$ and can be represented as the Mellin transform of an $h_p(M)$-admissible measure.*

Remarks. (i) Conjecture 1 extends the conjecture of Coates and Perrin-Riou [3,4], where they have formulated such a conjecture if p is good ordinary for M. In this case, in particular, the p-Newton and Hodge polygons coincide. (ii) The condition in part (iii) of Conjecture 1 is called the *condition of Dąbrowski–Panchishkin* (see also [16]). Here is an example where $P_{N,p}(d^+,M) = P_H(d^+,M)$, but $P_{N,p}(u,M) \not\equiv P_H(u,M)$: $M = M(f)\otimes M(g)$, where f, g are elliptic cusp forms of weights $w(f) > w(g)$ and where p is ordinary for f but supersingular for g. (iii) Conjecture 1 has been proved for Tate motive, and in the following cases: $M = \mathrm{Sym}^m M(f)$, $m = 1,2,3$ (see [1,18,11,6,2]), $M = M(f)\otimes M(g)$, $w(f) > w(g)$ (see [12]), and $M = M(f_1)\otimes M(f_2)\otimes M(f_3)$, $w(f_2)+w(f_3) > w(f_1)+1$ (see [2]).

3. Bounded p-adic L-functions of motives at supersingular primes

Assume, as before, that p is good for M, and that the inverse roots are indexed in such a way that $\mathrm{ord}_p\,\alpha_p^{(1)} \leqslant \mathrm{ord}_p\,\alpha_p^{(2)} \leqslant \cdots \leqslant \mathrm{ord}_p\,\alpha_p^{(d)}$. Let $L_p^{(\epsilon_0)}$ denote the corresponding p-adic L-function given by Conjecture 1. We can reformulate this conjecture in terms of power series in T, defining $L_p(M,\psi,T)$ as $L_p^{(\epsilon_0)}(\psi\chi_{(1+T)})$, where ψ is a fixed tame character such that $\psi(-1)=\epsilon_0$.

Let $\Phi_k(T)$ be the k-th cyclotomic polynomial. Fix a topological generator γ of $1+q\mathbb{Z}_p$, where $q=p$ for odd primes p, and $q=4$ for $p=2$. For any positive integer m, we define two power series in $\mathbb{Q}_p[[T]]$:

$$\log^+_{p,m}(T) := \frac{1}{p}\prod_{n=1}^{\infty}\left(\frac{\Phi_{p^{2n}}(\gamma^{-m}(1+T))}{p}\right), \qquad \log^-_{p,m}(T) := \frac{1}{p}\prod_{n=1}^{\infty}\left(\frac{\Phi_{p^{2n-1}}(\gamma^{-m}(1+T))}{p}\right).$$

The power series $\log_p^{\pm}(M,T) := \prod_{m=m_\star}^{m^\star}\log^{\pm}_{p,m}(T)$ are convergent on the open unit disc, and the only zeros of $\log_p^+(M,T)$ (resp. $\log_p^-(M,T)$) are simple zeros at $\gamma^m\zeta_{p^{2n}}-1$ (resp. $\gamma^m\zeta_{p^{2n-1}}-1$) for $m_\star \leqslant m \leqslant m^\star$ and $n \geqslant 1$, where ζ_{p^m} denotes a primitive p^m-th root of unity.

We say that a prime p is *very supersingular for M*, if it is good for M, $h_p(M) = \frac{m^\star - m_\star + 1}{2}$, and $\prod_{m=1}^{d^+}\alpha_p^{(m)} = -\prod_{m=1}^{d^+}\alpha_p^{(i_m)}$ for some other ordering of the inverse roots, still in such a way that $\mathrm{ord}_p\,\alpha_p^{(i_1)} \leqslant \mathrm{ord}_p\,\alpha_p^{(i_2)} \leqslant \cdots \leqslant \mathrm{ord}_p\,\alpha_p^{(i_d)}$. It corresponds to Pollack's condition $\alpha_p = -\alpha'_p$ in the case of modular forms.

Conjecture 2. *Assume that a prime p is very supersingular for M. Then* $L_p(M,\psi,T) = L_p^+(M,\psi,T)\cdot\log_p^+(M,T) + \prod_{i=1}^{d^+}\alpha_p^{(i)}\cdot L_p^-(M,\psi,T)\cdot\log_p^-(M,T)$, *where* $L_p^{\pm}(M,\psi,T)$ *are bounded.*

Theorem 1. *Conjecture* 1 *implies Conjecture* 2.

Proof. We imitate the proof of Theorem 5.1 in [17]. Define

$$G_\psi^+(M,T) := \frac{L_p(M,\psi,T)+L_p^\star(M,\psi,T)}{2}, \qquad G_\psi^-(M,T) := \frac{L_p(M,\psi,T)-L_p^\star(M,\psi,T)}{2\prod_{i=1}^{d^+}\alpha_p^{(i)}},$$

where $L_p^\star(M,\psi,T)$ denotes p-adic L-function corresponding to the second ordering of the inverse roots. The interpolation property from Conjecture 1 forces $G_\psi^+(M,\gamma^j\zeta_{p^{2n}}-1)=0$ and $G_\psi^-(M,\gamma^j\zeta_{p^{2n-1}}-1)=0$ for $m_\star \leqslant j \leqslant m^\star$ and $n>0$. Defining $L_p^{\pm}(M,\psi,T) := \frac{G_\psi^{\pm}(M,T)}{\log_p^{\pm}(M,T)}$, we are done. □

Remarks. (i) In a case $M=M(f)$ we obtain the plus/minus p-adic L-functions constructed by Pollack. Proof of Theorem 1 gives (unconditional) construction of plus/minus p-adic L-functions attached to $\mathrm{Sym}^m M(f)$ ($m=2,3$), $M(f)\otimes M(g)$, and $M(f_1)\otimes M(f_2)\otimes M(f_3)$ (see the end of Section 2). (ii) In a recent work by Lei, Loeffler and Zerbes [10], the authors generalize Pollack's decomposition for arbitrary modular forms in the very supersingular case and apply this to Iwasawa's Main Conjecture. (iii) Park and Shahabi [15], and Zhang [19] used the p-adic L-functions from [5] to construct plus/minus p-adic L-functions for Hilbert modular forms. (iv) There exists a variant of Conjecture 1 for motives over totally real number fields, and we can formulate a variant of Conjecture 2 for motives over totally real number fields as well [14]. (v) By a theorem of Elkies [9], there are infinitely many supersingular primes for a given elliptic curve defined over $\mathbb{Q}$, and hence for the corresponding newform of weight two. On the other hand, Lehmer's conjecture says that $\tau(n)\neq 0$ for any n, where $\Delta = \sum\tau(n)q^n$ denote the unique normalized newform of level one and weight 12.

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