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Partial Differential Equations

Traveling waves for a reaction–diffusion–advection system with interior or boundary losses

Étude d'un système de réaction–diffusion avec pertes

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ABSTRACT

This Note deals with the existence and qualitative properties of traveling wave solutions of a nonlinear reaction–diffusion system with losses inside the domain. In particular, we show the existence of a continuum of admissible speeds of traveling waves. Lastly, by considering losses concentrated near the boundary of the domain, these results are compared with those already known in the case of losses on the boundary.

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RÉSUMÉ

Cette Note a pour objet l'existence et les propriétés des solutions de type front progressif pour un système de réaction–diffusion non linéaire avec pertes à l'intérieur du domaine. Nous montrons en particulier l'existence d'un continuum de vitesses admissibles pour les fronts. Enfin, en considérant des pertes localisées près du bord, ces résultats sont comparés avec ceux déjà connus pour des pertes à la frontière du domaine.

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Version française abrégée

L'objectif de cette Note est l'étude des solutions de type front progressif d'un système de réaction–diffusion à deux équations de type proie–prédateur, avec la présence d'un terme de pertes. On considère le système suivant, posé dans un domaine cylindrique $\Omega = \mathbb{R}_x \times \omega_y \subset \mathbb{R}^N$, où ω est borné à bord régulier :

$$\begin{cases} U_t + \beta(y)U_x = \Delta u + f(y,U)V - h(y,U), \\ V_t + \beta(y)V_x = d^{-1}\Delta V - f(y,U)V, \end{cases} \tag{1}$$

avec des conditions de Neumann au bord. On suppose que le flot β est de classe $C^{0,\alpha}(\overline{\omega})$ et vérifie $\int_\omega \beta(y)\,\mathrm{d}y = 0$. Les fonctions f et h sont $C^{1,\alpha}(\overline{\omega} \times [0,+\infty);\mathbb{R})$, et du type KPP dans le sens suivant :

$$f(\cdot,0) = 0 < f(\cdot,U) \leqslant \frac{\partial f}{\partial U}(\cdot,0)U \quad \text{pour } U > 0, \quad \frac{\partial f}{\partial U} \geqslant 0, \quad f(\cdot,+\infty) = +\infty,$$

$$h(\cdot,0) = 0 \leqslant \frac{\partial h}{\partial U}(\cdot,0)U \leqslant h(\cdot,U) \leqslant KU \quad \text{où } K > 0,$$

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$$\int_\omega \frac{\partial h}{\partial U}(y,0)\,\mathrm{d}y > 0.$$

Le terme de pertes est ici la fonction h, qui agit à l'intérieur du domaine.

On s'intéresse d'abord dans cette Note aux fronts progressifs solutions non triviales de ce système, c'est-à-dire les solutions de la forme $U(t,x,y) = \tilde{U}(x-ct,y)$ et $V(t,x,y) = \tilde{V}(x-ct,y)$ avec $U > 0$, $0 < V < 1$ et les conditions à l'infini suivantes :

$$\begin{cases} \tilde{U}(+\infty,\cdot) = 0, \quad \tilde{V}(+\infty,\cdot) = 1, \\ \tilde{U}_x(-\infty,\cdot) = \tilde{V}_x(-\infty,\cdot) = 0. \end{cases} \tag{2}$$

Pour simplifier nos notations, on se place dans le repère avançant à vitesse c et on enlève les tildes.

On montre alors qu'il existe un continuum de vitesses admissibles. Plus précisément, si l'on introduit $\mu_{h,f}$ la valeur propre principale de (4), il existe une vitesse c^* telle qu'on a le théorème suivant :

Théorème 0.1. (a) *Supposons que* $\mu_{h,f}(0) < 0$. *Alors pour toute vitesse* $c > \max(0, c^*)$, *il existe un front progressif solution non triviale de* (1)–(2) *avec les conditions de Neumann sur* $\partial\Omega$.

(b) *Supposons que* $\sup_{\lambda\in\mathbb{R}}(\mu_{h,f}(\lambda) - \lambda^2) < 0$. *Alors* $c^* > 0$ *et il existe un front progressif avec vitesse* $c = c^*$ *solution non triviale de* (1)–(2) *avec les conditions de Neumann sur* $\partial\Omega$.

(c) *Réciproquement, soit* (U, V) *front progressif avec vitesse c solution de* (1)–(2) *avec les conditions de Neumann au bord, et tel que* $0 < U$ *et* $0 < V < 1$. *Alors U est borné,* $U(-\infty,\cdot) = 0$, $V(-\infty,\cdot) = V_\infty \in (0,1)$, $\mu_{h,f}(0) < 0$, $c > 0$ *et* $c \geqslant c^*$.

Dans [2,6], les auteurs ont étudié un système semblable (5)–(6), où les pertes ont lieu non pas à l'intérieur du domaine mais au bord, avec une condition de Robin $\frac{\partial U}{\partial n} + qU = 0$ sur $\partial\Omega$ où q constante positive. Il existe aussi pour ce système un continuum de vitesses admissibles de la forme $(\gamma_q^*, +\infty)$.

Théorème 0.2. *Soit* $(h_k)_{k\in\mathbb{N}}$ *une suite de fonctions linéaires en U et telles que* :

$$\begin{cases} \exists \varepsilon_k \searrow 0 \text{ tel que } \dfrac{\partial h_k}{\partial U}(\cdot,0) \to 0 \text{ uniformément sur } \omega\backslash\big(\partial\omega + \overline{B}(0,\varepsilon_k)\big), \\ \varepsilon_k \left\| \dfrac{\partial h_k}{\partial U}(\cdot,0) \right\|_{L^\infty(\omega)} = O(1), \\ g(\sigma) := \displaystyle\int_0^1 \varepsilon_k \frac{\partial h_k}{\partial U}\big(\sigma - \varepsilon_k s n(\sigma), 0\big)\,\mathrm{d}s \to q \text{ uniformément pour } \sigma \in \partial\omega \text{ et où } n \text{ normale de } \partial\omega. \end{cases}$$

Alors la suite $(c_k^*)_k$ *des vitesses minimales pour le système* (1) *avec* $h = h_k$ *converge vers* γ_q^* *la vitesse minimale pour le système* (5) *avec conditions au bord* (6). *De plus, en dimension* $d = 2$, *il existe une suite de fronts progressifs solutions de* (1)–(3) *et* (2) *avec* $h = h_k$ *et* $c > \max(0, \gamma_q^*)$, *qui converge faiblement dans* $H^1_{loc}(\overline{\Omega})$ *vers un front progressif non trivial solution de* (5)–(6) *et* (2).

1. Introduction

Reaction–diffusion equations and related spreading phenomena have been extensively studied in the past decades, due to the numerous applications in various fields of natural sciences, ranging from chemical and biological contexts to combustion and many-particle systems (see [1,3,7,8] for reviews of this mathematical area). However, much less is known for reaction–diffusion systems, where several unknowns are involved. Indeed, previous studies for systems have been mainly limited to competition or cooperation–diffusion systems, for which the comparison principle holds. Very little is known rigorously in the case of combustion or prey-predator systems in heterogeneous media or in higher dimensions.

We consider here such a problem in a cylindrical domain $\Omega = \mathbb{R}_x \times \omega_y \subset \mathbb{R}^N$, where ω is smooth and bounded:

$$\begin{cases} U_t + \beta(y)U_x = \Delta u + f(y,U)V - h(y,U), \\ V_t + \beta(y)V_x = d^{-1}\Delta V - f(y,U)V, \end{cases}$$

with Neumann boundary conditions

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = 0 \quad \text{on } \partial\Omega, \text{ where } n \text{ denotes the outward unit normal on } \partial\omega. \tag{3}$$

We assume that the flow β does not depend on x, is of class $C^{0,\alpha}(\overline{\omega})$ and is such that $\int_\omega \beta(y)\,\mathrm{d}y = 0$. The functions f and h are assumed to be in $C^{1,\alpha}(\overline{\omega}\times[0,+\infty);\mathbb{R})$. Moreover, the reaction term f satisfies:

$$f(\cdot,0) = 0 < f(\cdot,U) \leqslant \frac{\partial f}{\partial U}(\cdot,0)U \quad \text{for } U > 0, \quad \frac{\partial f}{\partial U} \geqslant 0, \quad f(\cdot,+\infty) = +\infty,$$

and the loss term h, which can take place in the whole domain, is such that:

$$h(\cdot,0) = 0 \leqslant \frac{\partial h}{\partial U}(\cdot,0)U \leqslant h(\cdot,U) \leqslant KU \quad \text{where } K > 0, \quad \int_\omega \frac{\partial h}{\partial U}(y,0)\,\mathrm{d}y > 0.$$

By analogy with the single equation case, those are KPP type hypotheses which allow to use comparisons with the linearized problem. The positivity of $\int_\omega \frac{\partial h}{\partial U}(y,0)\,\mathrm{d}y$ means that the loss is nontrivial.

We take interest in this Note in the nontrivial traveling wave solutions of system (1), that is, solutions of the form $U(t,x,y) = \tilde{U}(x-ct,y)$ and $V(t,x,y) = \tilde{V}(x-ct,y)$ with $U > 0$, $0 < V < 1$, which satisfy (1) with Neumann boundary conditions and:

$$\begin{cases} \tilde{U}(+\infty,\cdot) = 0, \quad \tilde{V}(+\infty,\cdot) = 1, \\ \tilde{U}_x(-\infty,\cdot) = \tilde{V}_x(-\infty,\cdot) = 0. \end{cases}$$

To simplify our notations, we will place ourselves in the moving frame with speed c and drop the tildes. System (1) then becomes a time-independent elliptic problem.

As announced, we will first give some necessary and sufficient conditions on the existence of such solutions. In particular, we will show the existence of a minimal admissible speed. This result is directly related to the linearized problem, as we explain below. Lastly, we show how this model with losses distributed inside the domain can be related to the model with losses concentrated on the boundary.

2. Study of the system with losses distributed inside the domain

Here, we give some results about the existence of traveling wave solutions of (1)–(3) and (2).

2.1. Preliminary linear analysis

From the KPP hypotheses, we expect the behavior of the system to be determined by the linearized system with $U = 0$ and $V = 1$. Thus, we introduce the following principal eigenvalue problem, depending on a parameter $\lambda \in \mathbb{R}$:

$$\begin{cases} -\Delta_y \phi_\lambda - \lambda u(y)\phi_\lambda + \left(\frac{\partial h}{\partial U}(y,0) - \frac{\partial f}{\partial U}(y,0)\right)\phi_\lambda = \mu_{h,f}(\lambda)\phi_\lambda & \text{in } \omega, \\ \frac{\partial \phi_\lambda}{\partial n} = 0 & \text{on } \partial\omega, \\ \phi_\lambda > 0 & \text{in } \overline{\omega}. \end{cases} \tag{4}$$

One can check that the function $\mu_{h,f}$ is C^1 and concave with respect to $\lambda \in \mathbb{R}$. By looking at solutions of the form $U(x,y) = \phi_\lambda(y)e^{-\lambda x}$ of the linearized system, we obtain the condition $\mu_{h,f}(\lambda) = \lambda^2 - c\lambda$. When $\mu_{h,f}(0) > 0$, by concavity with respect to λ, this condition admits a negative solution and $U = 0$ would then be stable for the linearized problem. Here, we want U to be decreasing toward 0 on $+\infty$, see (2). Thus, we will assume that $\mu_{h,f}(0) < 0$, and we can then define

$$c^* = \min\{c \in \mathbb{R} / \exists \lambda > 0 \text{ such that } \mu_{h,f}(\lambda) = \lambda^2 - c\lambda\}.$$

We will see that c^* is indeed the minimal speed of the traveling wave solutions.

2.2. Existence of traveling wave solutions

Our first main result is the existence of traveling wave solutions for any speed $c > \max(0, c^*)$. More precisely, we have the following theorem:

Theorem 2.1. (a) *Assume that* $\mu_{h,f}(0) < 0$. *For any* $c > \max(0, c^*)$, *there exists a nontrivial traveling wave solution* (U, V) *of* (1)–(2) *with Neumann boundary conditions on* $\partial\Omega$.

(b) *Assume that* $\sup_{\lambda\in\mathbb{R}}(\mu_{h,f}(\lambda) - \lambda^2) < 0$. *Then* $c^* > 0$ *and there exists a nontrivial traveling wave solution* (U, V) *of* (1)–(2) *with Neumann boundary conditions on* $\partial\Omega$ *and minimal speed* $c = c^*$.

In the case $c > c^*$, the proof given in [4] relies on the construction of sub- and super-solutions:

- $\overline{V} = 1$ and $\underline{V}$ is of the form $\max(0, 1 - \psi(y)e^{-\delta x})$ with $\psi > 0$ and δ a small positive constant;
- $\overline{U}(x,y) = \phi_\lambda(y)e^{-\lambda x}$ and $\underline{U}(x,y) = \max(0, \phi_\lambda(y)e^{-\lambda x} - \gamma\phi_{\lambda+\eta}(y)e^{-(\lambda+\eta)x})$, where λ is the (positive) smallest real number such that $\mu_{h,f}(\lambda) = \lambda^2 - c\lambda$, and $\eta > 0$ is small and $\gamma > 0$ is large.

We then use a fixed point theorem on truncated cylinders, and conclude by passing to the limit to obtain global solutions.

The case $c = c^*$ is more complicated, as we can't find a subsolution for U by proceeding with the above method. Thus, we consider a sequence (U_n, V_n) of traveling wave solutions with speed $c_n \to c^*$. The main step is to locate the interface, that is to find a sequence $(\tilde{x}_n, \tilde{y}_n)$ such that $\inf_n |\nabla V_n(\tilde{x}_n, \tilde{y}_n)| > 0$. This can be done using a lemma on the behavior of V behind the front, which states that $V(-\infty, \cdot) < a^* < 1$ for some a^* independent of the speed c. It can then be shown, using standard estimates, that $(U_n(\tilde{x}_n + \cdot, \cdot), V_n(\tilde{x}_n + \cdot, \cdot))$ converges to a nontrivial traveling wave solution with minimal speed c^*.

This result is a generalization of what is already known in the KPP single equation case. It shows that under an instability assumption on the linearized problem with $U = 0$ and $V = 1$, traveling waves exist for a continuum of admissible speeds.

Conversely to the previous theorem, we also show that the condition $c \geqslant c^*$ is necessary, as well as the instability assumption $\mu_{h,f}(0) < 0$ on the linearized problem.

Theorem 2.2. *Let (U, V) be a traveling wave solution with speed c of (1)–(2), and such that $0 < U$ and $0 < V < 1$. Then U is bounded, $U(-\infty, \cdot) = 0$, $V(-\infty, \cdot) = V_\infty \in (0, 1)$, $\mu_{h,f}(0) < 0$, $c > 0$ and $c \geqslant c^*$.*

The proof of this theorem relies on the integration of the equations verified by U and V, which gives us some estimates, and the existence of the limits $U(-\infty, \cdot)$ and $V(-\infty, \cdot)$. We can then use the behavior of U near $+\infty$, where the problem is almost linear, to get the conditions on $\mu_{h,f}(0)$ and the speed c.

3. Consistency with the model with losses on the boundary

We introduce the following system, studied in [2,6]:

$$\begin{cases} U_t + \beta(y)U_x = \Delta U + f(y, U)V, \\ V_t + \beta(y)V_x = d^{-1}\Delta V - f(y, U)V, \end{cases} \tag{5}$$

with boundary conditions

$$\begin{cases} \dfrac{\partial U}{\partial n} + qU = 0, \\ \dfrac{\partial V}{\partial n} = 0, \end{cases} \quad \text{on } \partial\Omega, \tag{6}$$

where q is a positive constant. Here, the loss does not take place inside the domain but along the boundary, hence the Robin boundary condition on U. The study of this problem presents similar results to those described in Section 2. In particular, there is also a continuum of admissible speeds of the form $(\gamma_q^*, +\infty)$ for the existence of traveling wave solutions. Since both models are physically relevant (in population dynamics, the saturation can induce a larger death rate inside the domain, or make the individuals leave it), it is natural to consider losses distributed inside the domain and converging toward a Dirac mass on the boundary, and then look at the convergence of the related problems and solutions.

We consider a sequence $(h_k)_{k \in \mathbb{N}}$ such that:

$$\begin{cases} \exists \varepsilon_k \searrow 0 \text{ such that } \dfrac{\partial h_k}{\partial U}(\cdot, 0) \to 0 \text{ uniformly in } \omega \backslash \big(\partial\omega + \overline{B}(0, \varepsilon_k)\big), \\ \varepsilon_k \left\| \dfrac{\partial h_k}{\partial U}(\cdot, 0) \right\|_{L^\infty(\omega)} = O(1), \\ g(\sigma) := \displaystyle\int_0^1 \varepsilon_k \frac{\partial h_k}{\partial U}\big(\sigma - \varepsilon_k s n(\sigma), 0\big)\,\mathrm{d}s \to q \text{ uniformly in } \sigma \in \partial\omega. \end{cases} \tag{7}$$

Let us note c_k^* and γ_q^* the minimal speeds related respectively to systems (1)–(3)–(2) with $h = h_k$ and (5)–(6)–(2). We assume that they are well defined, which is the case for c_k^* as long as $\mu_{h_k,f}(0) < 0$, as seen above. Similarly, it has been shown in [2,6] that γ_q^* is well defined under the same assumption on the principal eigenvalue of the linearized problem with losses on the boundary, and that then there exist traveling front solutions for any speed $c > \gamma_q^*$.

Theorem 3.1. *Under the above assumptions, we have that $c_k^* \to \gamma_q^*$ as $k \to +\infty$.*

To ensure the accordance between the two models, we want to show the convergence of the traveling wave solutions of (1)–(3) and (2) with $h = h_k$ and some speed $c > \gamma_q^*$ to a traveling wave solution of (5)–(6) and (2) with the same speed. The main difficulty is the lack of bounds on h_k, and thus the lack of estimates and of a Harnack inequality. Here, we will only consider particular solutions which satisfy some exponential bounds from below and above, in order to overcome this difficulty.

Theorem 3.2. *Let $(h_k)_{k\in\mathbb{N}}$ be a sequence of functions linear with respect to U and such that* (7) *is satisfied. Let also $(U_k, V_k)_{k\in\mathbb{N}}$ be a sequence of nontrivial traveling wave solutions with speed $c > \gamma_q^*$ of problem* (1)–(3) *and* (2) *with $h = h_k$, and let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers such that for any $k \in \mathbb{N}$, we have*:

$$\lambda_k^2 - c\lambda_k = \mu_{h_k}(\lambda_k).$$

We assume that there exist $0 < \Lambda_1 < \Lambda_2$ and $C_1, C_2, C_3 > 0$ such that for all $k \in \mathbb{N}$ and $(x, y) \in \overline{\Omega}$:

$$U_k(x, y) < C_1 e^{-\lambda_k x}, \qquad \max\left(0, C_2 e^{-\Lambda_1 x} - C_3 e^{-\Lambda_2 x}\right) < U_k(x, y). \tag{8}$$

Then up to extraction of a subsequence, (U_k, V_k) converges weakly in $H^1_{loc}(\overline{\Omega})$ to a nontrivial solution (U, V) of problem (5)–(6) *and* (2).

The bounds (8) are proved to hold in dimension $d = 2$. Indeed, we know from the proof of Theorem 2.1 that those bounds are satisfied for each k and $c > c_k^*$. To make them independent of $k \in \mathbb{N}$, we have to use strong estimates on the principal eigenfunctions of (4) which hold only in dimension 2 ($d = 2$), where $H^1(\omega)$-estimates imply $C^{0,1/2}(\omega)$-estimates. The discussion above leads to the following corollary of Theorem 3.2:

Corollary 3.3. *Let $(h_k)_{k\in\mathbb{N}}$ be a sequence of functions linear with respect to U and such that* (7) *is satisfied. In dimension $d = 2$, up to extraction of some subsequence, there exists a sequence of traveling wave solutions of problem* (1)–(3) *and* (2) *with $h = h_k$ and $c > \max(0, \gamma_q^*)$, that converges weakly in $H^1_{loc}(\overline{\Omega})$ to a nontrivial traveling wave solution (U, V) of problem* (5)–(6) *and* (2).

The dedicated proofs are given in [5].

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