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Topology/Dynamical Systems

A billiard containing all links

Un billard réalisant tout entrelacs

Pierre Dehornoy

UMPA, Ens de Lyon, 46, allée d'Italie, 69364 Lyon, France

A R T I C L E   I N F O

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A B S T R A C T

We construct a 3-dimensional billiard realizing all links as collections of isotopy classes of periodic orbits. For every branched surface supporting a semi-flow, we construct a 3*d*-billiard whose collections of periodic orbits contain those of the branched surface. R. Ghrist constructed a knot-holder containing any link as collection of periodic orbits. Applying our construction to his example provides the desired billiard.

R É S U M É

On construit un billard tridimensionnel réalisant tout entrelacs fini comme collection d'orbites périodiques. Plus généralement, étant donné un patron, c'est-à-dire une surface branchée munie d'un semi-flot, on construit un billard dont la collection des orbites périodiques contient celle du patron. R. Ghrist a construit un tel patron contenant tous les entrelacs. On obtient le billard souhaité en appliquant notre construction à son exemple.

For every compact domain of $\mathbb{R}^3$ with a smooth boundary, one can play *billiard* inside it, with the rule that rays reflect perfectly on the boundary, see [6]. If the boundary has corners, the reflection is not defined, and we only consider orbits avoiding them. Thus a periodic orbit with no self-intersection point yields a knot in $\mathbb{R}^3$ and one can wonder about the relation between the shape of the billiard and the knots arising in this way. In a cube, the latter correspond to the so-called *Lissajous knots*, see [2]. In a cylinder the situation is more intricate, see [3]. It is asked in [4] and [5] whether there exists a billiard containing all knots as periodic orbits. In this Note, we provide a positive answer to this question.

Definition 1. A *template* (see Figs. 1, 2) is a smooth compact surface S with boundary embedded in $\mathbb{R}^3$ and equipped with a non-vanishing vector field V so that:

(i) V is tangent to the surface S and to its boundary,
(ii) there exist finitely many branching segments — called *convergence segments* — transverse to V, where three pages P^+, P^- and P^o of the surface meet, with V leaving P^+ and P^- and entering P^o,
(iii) there exist finitely many branching points — called *separation points* — on the boundary of S whose neighborhood is diffeomorphic to an open disc cut along the bottom vertical radius and equipped with the top–bottom vector field.

E-mail address: pierre.dehornoy@umpa.ens-lyon.fr.

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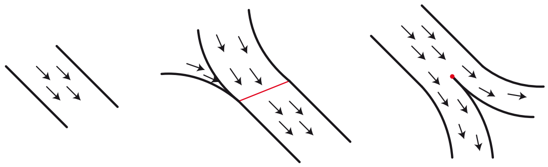

Fig. 1. How does a template look like. A generic point, a convergence segment and a separation point.

Fig. 1. Ce à quoi ressemble un patron. Un point générique, un segment de convergence et un point de séparation.

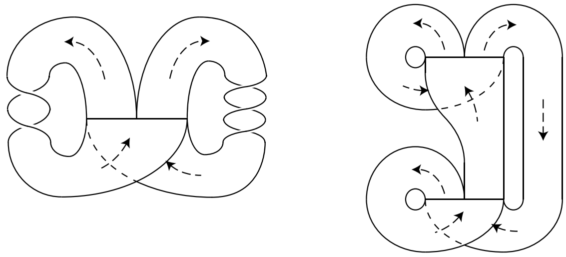

Fig. 2. A generalized Lorenz' template, and the Ghrist's template [1].

Fig. 2. Un patron de Lorenz généralisé et le patron de Ghrist [1].

First suppose we are given a template T_0 that can be immersed into the plane (for example Ghrist's template works, but not the generalized Lorenz').[1] Distort it into T_1 in such a way that the projection of T_1 on the horizontal plane is obtained by gluing ribbons with slopes $(\pm 1, 0)$ or $(0, \pm 1)$ for generic points, isosceles-rectangular triangles for changes of direction, ribbons with slope $(0, -1)$ for convergence segments, and parabolic church shapes. These patterns are depicted in Figs. 3 and 4. Note that they fit well into the integer lattice $\mathbb{Z}^2$. Now we associate a billiard B_1 to T_1 by lifting it in $\mathbb{R}^3$ so that ribbons are 1 unit thick along the vertical direction and match with each other. Convergence segments and separation points deserve a special treatment depicted in Figs. 5 and 6. There could be level gaps, but these can be settled using vertical double bends, see Fig. 7.

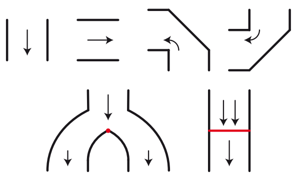

Fig. 3. Patterns for the template T_1.

Fig. 3. Pièces du patron T_1.

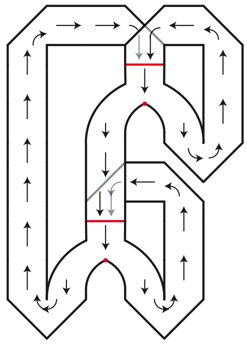

Fig. 4. Ghrist's template distorted into a T_1-like template.

Fig. 4. Le patron de Ghrist déformé pour être de type T_1.

Our claim is the following:

Theorem 2. *For every template T_0 embedded in $\mathbb{R}^3$, every finite collection of periodic orbits of T_0 is isotopic to a finite collection of periodic orbits of the billiard B_1 constructed above.*

Applying this to Ghrist's template [1] yields

Corollary 3. *There exists a domain in $\mathbb{R}^3$ with a piecewise smooth boundary* (*see Fig.* 9) *so that any link appears as a family of periodic billiard trajectories.*

Proof. We only prove the theorem for knots, the case of links being similar.

Let γ_0 be a periodic orbit on T_0. Since the templates T_0 and T_1 are isotopic, there exists a periodic orbit γ_1 of T_1 isotopic to γ_0. Let p be an arbitrary point on γ_1. One associates an infinite periodic word $w^{\mathbb{N}}_{\gamma_1,p}$ on the alphabet $\{0, 1\}$ so that when one follows γ_1, the sequence of left/right-choices at separation points is described by the letters of $w^{\mathbb{N}}_{\gamma_1,p}$.

Let q be a point inside B_1, and suppose that it projects to T_1 on a point where the flow is parallel to the y-direction. Call (x_q, y_q) the horizontal coordinates of q and z_q the vertical one (along which the projection is performed). Playing billiard in B_1 along the y-direction does not change (x_q, z_q), unless one crosses corners, churches or convergence boxes. In the first

[1] For orientable templates, the so-called *bell trick* can do the job.

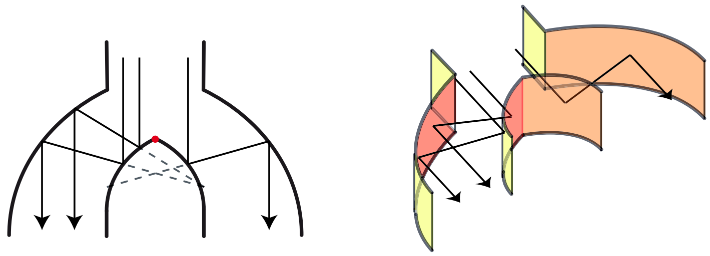

Fig. 5. How to realize a separation point using billiards: on the left side a horizontal cut. The left two curves are confocal parabolae, the exterior one being homothetic to the interior one by a factor 2. In this way, vertical entering rays go out vertically and their mutual distances are doubled. The same thing holds on the right of the separation point.

Fig. 5. Réalisation d'un point de séparation par un billard. Les deux courbes de gauche sont des paraboles confocales, l'externe étant homothétique de l'interne par un facteur 2. Ainsi les trajectoires arrivant verticalement par le haut sortent verticalement et leurs distances mutuelles sont doublées. La même chose se produit à droite du segment de séparation.

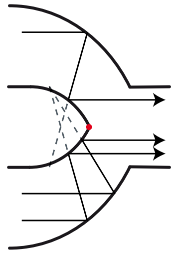

Fig. 6. How to realize a convergence segment by billiards: a vertical cut. The shape is the same as for separation points, but the flow is reversed, and the shape turned 90° along the y-direction.

Fig. 6. Réalisation d'un segment de convergence. On prend la même pièce que pour les points de séparation, en la tournant verticalement. Le flot est également renversé.

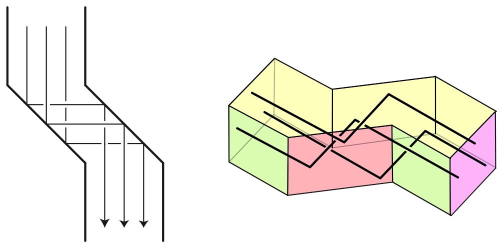

Fig. 7. A horizontal double bend. At each corner some crossings which were not on the template T_0 may appear. Nevertheless, since corners come in pairs, these extra-crossings disappear with a Reidemeister-*II*-move. Level gaps are solved with this pattern turned vertically.

Fig. 7. Une chicane horizontale. À chaque coin apparaissent des croisements qui n'existaient pas sur le patron T_0. Néanmoins, comme les coins viennent toujours par paires, ces croisements supplémentaires aussi, et un mouvement de Reidemeister les supprime.

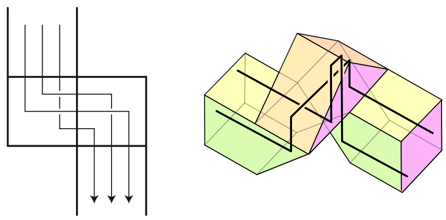

Fig. 8. How to realize a half-twist. Note on the left picture that the projections of any two strands cross exactly once, and that horizontal strands go above vertical ones. Since a braid in which any two strands cross exactly once and positively is half-turn Δ_n, this billiard realizes a half-twist.

Fig. 8. Réalisation d'un demi-tour. Comme indiqué sur la partie gauche, les projections horizontales de deux trajectoires se coupent exactement une fois, avec le brin O–E par au-dessus du brin N–S. Or une tresse dont deux brins quelconques se coupent exactement une fois et positivement est isotope au demi-tour Δ_n.

case, the restrictions we imposed on possible shapes force a second corner to follow the first one, and the x, z-coordinates mod 1 are not affected by two consecutive changes of direction. When crossing a church, the x-coordinate is doubled mod 1. Similarly, crossing a convergence box backwards doubles the z-coordinate. In other words, if we play billiard along the y-direction, the future is encoded in the x-coordinate, while the past is encoded in the z-coordinate. In particular, if q lies on a periodic orbit in the y-direction, the dyadic expansions of x_q mod 1 and z_q mod 1 are both periodic and the associated patterns are mirrors one of the other.

It is therefore natural to compare the orbit γ_1 on T_1 passing through p and the orbit $\tilde{\gamma}_1'$ on B_1 passing through $(0.w^{\mathbb{N}}_{\gamma_1,p}, 0, 0.\bar{w}^{\mathbb{N}}_{\gamma_1,p})$ and going along the y-direction. Since $\tilde{\gamma}_1'$ is horizontal except in convergence boxes, it is the lift of a periodic orbit γ_1' in the planar template T_1 considered as a planar billiard. Therefore a knot-diagram of $\tilde{\gamma}_1'$ is obtained from γ_1' by removing the ambiguities at crossings.

A crossing of γ_1' may arise in three situations only. Either it arises in a corner with two strands in the same box, in which case the previous-or-next corner provides another crossing for the same pair of strands, so that the pair will disappear with a Reidemeister-*II*-move, see Fig. 7. Or it arises when different ribbons cross, in which case the same ribbons cross in T_1. Or it arises at a corner when two ribbons become parallel just before a convergence box, in which case the crossing already exists in γ_1 when the two ribbons of T_1 overlap. Therefore, the horizontal projection of $\tilde{\gamma}_1'$ can be distorted to γ_1 using Reidemeister-*II*-moves only, and so the two knots are isotopic.

We still have to address the case of a half-twist on a non-orientable template. This can be fixed with the billiard of Fig. 8. So the proof is complete. $\square$

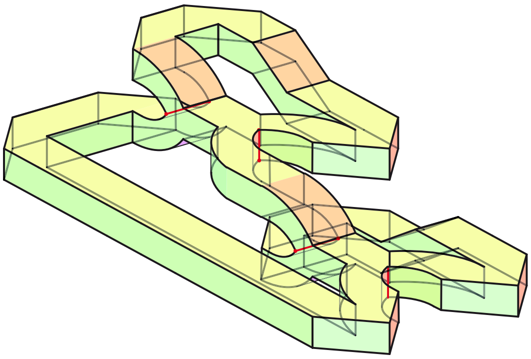

Fig. 9. The billiard associated to Ghrist's template.

Fig. 9. Le billard associé au patron de Ghrist.

Note that our construction can be smoothed so that the boundary of the billiard becomes a smooth surface. On the other hand, the parabolae are crucial in order to double the coordinates, preventing us to construct a billiard with piecewise-linear boundary. We are left with these two questions:

(i) Is it possible to construct a polygonal billiard containing all links as periodic orbits?
(ii) Is it possible to construct a convex billiard containing all links as periodic orbits?

Acknowledgements

I thank J. O'Rourke, the website *MathOverflow* and all participants to the conversation [5] for raising this question.

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