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The Fourier–Stieltjes transform of Minkowski's $?(x)$ function and an affirmative answer to Salem's problem

La transforme de Fourier–Stieltjes de la fonction $?(x)$ de Minkowski et une réponse positive au probléme de Salem

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ABSTRACT

By using structural and asymptotic properties of the Kontorovich–Lebedev transform associated with Minkowski's question mark function, we give an affirmative answer to the question posed by R. Salem (Trans. Amer. Math. Soc. 53 (3) (1943) 439) whether its Fourier–Stieltjes transform vanishes at infinity.

RÉSUMÉ

Grace à des propriétés structurelles et asymptotiques de la transformation de Kontorovich–Lebedev associé à la fonction point d'interrogation de Minkowski, on apporte une réponse positive à la question posée par R. Salem (Trans. Amer. Math. Soc. 53 (3) (1943) 439): la transformée de Fourier–Stieltjes de la fonction point d'interrogation de Minkowski est-elle nulle à l'infini?

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1. Introduction and auxiliary results

We deal here with the so-called Minkowski question mark function $?(x) : [0, 1] \mapsto [0, 1]$, which is defined by [2]

$$?\big([0, a_1, a_2, a_3, \ldots]\big) = 2\sum_{i=1}^{\infty}(-1)^{i+1}2^{-\sum_{j=1}^{i} a_j}, \tag{1}$$

where $x = [0, a_1, a_2, a_3, \ldots]$ stands for the representation of x by a regular continued fraction. We note that despite the symbol $?(x)$ is quite odd to denote a function in such a way, we mildly resist the temptation of changing the notation, which was used in the original Salem's paper [11]. It is well known that $?(x)$ is continuous, strictly increasing and singular with respect to Lebesgue measure. It can be extended on $[0, \infty)$ by using the following functional equations:

$$?(x) = 1 - ?(1 - x), \quad 0 \leqslant x \leqslant 1, \tag{2}$$

$$?(x) = 2?\left(\frac{x}{x+1}\right), \quad x \geqslant 0, \tag{3}$$

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$$?(x)+?\left(\frac{1}{x}\right)=2, \quad x>0. \tag{4}$$

This function decreases exponentially near the origin

$$?(x)=O\left(2^{-1/x}\right), \quad x\to 0. \tag{5}$$

Key values are $?(0)=0$, $?(1)=1$. Some properties of its moments can be found in [1]. As usual, we define the finite Fourier–Stieltjes transform of the Minkowski question mark function by the following Stieltjes integral:

$$F(t)=\int_0^1 e^{ixt}\,\mathrm{d}?(x), \quad t\in\mathbb{R}. \tag{6}$$

In 1943 Salem asked [11] whether $F(t)\to 0$ as $|t|\to\infty$, putting the question for the corresponding Fourier–Stieltjes coefficients. We note, that the question to determine whether a given measure is a *Rajchman measure* (that is, whose Fourier transform vanishes at infinity), as far as measures arising from singular monotone functions are concerned, is a very delicate question. This situation is quite different from the one when the measure is absolutely continuous and the classical Riemann–Lebesgue lemma for the class L_1 can be applied (cf. [13], Ch. IV). For singular measures there are various examples and the Fourier–Stieltjes transform need not tend to zero, although there do exist measures for which it goes to zero. For instance, Salem [11,12] gave examples of singular functions, which are strictly increasing and whose Fourier coefficients still do not vanish at infinity. On the other hand, Menchoff in 1916 [7] gave a first example of a singular distribution whose coefficients vanish at infinity. Wiener and Wintner [15] (see also [4]) proved in 1938 that for every $\varepsilon>0$ there exists a singular monotone function such that its Fourier coefficients behave as $n^{-\frac{1}{2}+\varepsilon}$, $n\to\infty$.

Our aim in this Note is to solve this Salem's problem. In fact, to achieve this goal several attempts were made by the author, involving various representations of the Fourier–Stieltjes transform (6) via the composition of different integral transformations, such as the Laplace, Hankel transforms and Riemann–Liouville fractional integro-differential operators (cf. [18]). But finally we came to the conclusion that an application of index transforms [16] will be a rather effective tool to achieve the goal and give an affirmative answer to the question. Namely, we will explore composition and asymptotic properties of the Kontorovich–Lebedev transform with respect to the Haar measure $\frac{\mathrm{d}x}{x}$ (see in [5,14,8,16,18,17])

$$KL(\tau)=\int_0^\infty K_{i\tau}(x)f(x)\frac{\mathrm{d}x}{x}, \quad \tau\in\mathbb{R}_+, \tag{7}$$

considering f as a continuous function on $\mathbb{R}_+$ having a suitable behaviour at infinity and near zero. The kernel of this transform $K_{i\tau}(x)$ is the modified Bessel function (cf. [3], vol. II) of the pure imaginary index $i\tau$. In the sequel we will need an asymptotic behaviour at infinity of the Kontorovich–Lebedev transform (7) and its modification involving a shift of the argument of the modified Bessel function. A basic asymptotic expansion of this type was given in [8] by using a composition representation of the Kontorovich–Lebedev transform in terms of the Laplace and Fourier transforms (cf. from [14,18]) and corresponding asymptotic expansions developed in [9] and [6]. Precisely, if f is continuous for $x>0$, behaves as $O(x^b)$, $b>0$ as $x\to 0$ and possesses the asymptotic expansion

$$f(x)\sim e^{x\cos\beta}\sum_{n=0}^\infty a_n x^{-n}, \quad x\to+\infty,\ 0<\beta\leqslant\frac{\pi}{2}, \tag{8}$$

then for each $N\geqslant 1$

$$KL(\tau)=2^{1/2}\pi^{3/2}e^{-\pi\tau}\sum_{n=0}^{N-1}a_n(\sin\beta)^{n+1/2}P^{-n-1/2}_{i\tau-1/2}(-\cos\beta)+O\left(e^{-\beta\tau}\tau^{-N}\right), \quad \tau\to+\infty, \tag{9}$$

where $P_\nu^\mu(z)$ is associated Legendre functions or conical functions [3], vol. I.

The key ingredient for our proof will be the following integral with respect to an index of the modified Bessel function:

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^\infty \tau e^{\lambda\tau}\,\left(t+\left(1+t^2\right)^{1/2}\right)^{i\tau}K_{i\tau}(x)\,\mathrm{d}\tau\\&\quad=x\exp\left(-x\left[\left(1+t^2\right)^{1/2}\cos\lambda-it\sin\lambda\right]\right)\left[\left(1+t^2\right)^{1/2}\sin\lambda+it\cos\lambda\right], \quad x,t>0\end{aligned} \tag{10}$$

and $0\leqslant\lambda<\frac{\pi}{2}$. It can be deduced, for instance, employing relation (2.16.48.20) in [10] and making differentiation by a parameter. Finally in this section, for further use let us return to (6) and integrate by parts in the Stieltjes integral subtracting a simple rational function. Thus we derive

$$F(t)=\int_0^1 e^{ixt}\,\mathrm{d}\left[?(x)-\frac{2x}{1+x}\right]+2\int_0^1 \frac{e^{ixt}}{(1+x)^2}\,\mathrm{d}x=F_1(t)+O\left(\frac{1}{t}\right),\quad t>1, \tag{11}$$

where

$$F_1(t)=-it\int_0^1 e^{ixt}\left[?(x)-\frac{2x}{1+x}\right]\mathrm{d}x. \tag{12}$$

2. A solution to Salem's problem

Our main result can be formulated as follows:

Theorem 2.1. *Let $t\in\mathbb{R}$. Then $\int_0^1 e^{ixt}\,\mathrm{d}?(x)=o(1)$, $|t|\to\infty$.*

Proof. Without loss of generality we prove the theorem for positive t. So our goal is to estimate $F_1(t)$ given by (12) when $t\to+\infty$. First we pass to the limit through equality (10) when $\lambda\to\frac{\pi}{2}-$. This yields

$$\frac{1}{\pi}\lim_{\lambda\to\frac{\pi}{2}-}\int_{-\infty}^{\infty}\tau e^{\lambda\tau}\,\left(t+\left(1+t^2\right)^{1/2}\right)^{i\tau}K_{i\tau}(x)\,\mathrm{d}\tau=x\left(1+t^2\right)^{1/2}e^{ixt},\quad x,t>0. \tag{13}$$

Hence we write $F_1(t)$ in the form

$$F_1(t)=\frac{t}{\pi i\left(1+t^2\right)^{1/2}}\int_0^1\left[\frac{?(x)}{x}-\frac{2}{1+x}\right]\lim_{\lambda\to\frac{\pi}{2}-}\int_{-\infty}^{\infty}\tau e^{\lambda\tau}\,\left(t+\left(1+t^2\right)^{1/2}\right)^{i\tau}K_{i\tau}(x)\,\mathrm{d}\tau\,\mathrm{d}x. \tag{14}$$

But since for each $x,t>0$ and $0\leqslant\lambda<\frac{\pi}{2}$ (see (10)) $|\int_{-\infty}^{\infty}\tau e^{\lambda\tau}(t+(1+t^2)^{1/2})^{i\tau}K_{i\tau}(x)\,\mathrm{d}\tau|\leqslant x[t+(1+t^2)^{1/2}]$ and $\int_0^1|?(x)-\frac{2x}{1+x}|\,\mathrm{d}x\leqslant 1+2\int_0^1\frac{x\,\mathrm{d}x}{1+x}=3-\log 4$, we can take out the limit in (14) having the representation

$$F_1(t)=\frac{t}{\pi i\left(1+t^2\right)^{1/2}}\lim_{\lambda\to\frac{\pi}{2}-}\int_0^1\left[\frac{?(x)}{x}-\frac{2}{1+x}\right]\int_{-\infty}^{\infty}\tau e^{\lambda\tau}\left(t+\left(1+t^2\right)^{1/2}\right)^{i\tau}K_{i\tau}(x)\,\mathrm{d}\tau\,\mathrm{d}x. \tag{15}$$

Our goal now is to invert the order of integration in (15). To do this we employ the uniform inequality for the modified Bessel function (cf. [5,18])

$$\left|K_{i\tau}(x)\right|\leqslant\frac{x^{-1/4}}{\sqrt{\sinh\pi\tau}},\quad x,\tau>0 \tag{16}$$

and asymptotic property (5) of the Minkowski question mark function near the origin. Consequently,

$$\int_0^1\left|\frac{?(x)}{x}-\frac{2}{1+x}\right|\int_{-\infty}^{\infty}\left|\tau e^{\lambda\tau}\left(t+\left(1+t^2\right)^{1/2}\right)^{i\tau}K_{i\tau}(x)\right|\mathrm{d}\tau\,\mathrm{d}x$$
$$\leqslant\int_0^1\left|\frac{?(x)}{x}-\frac{2}{1+x}\right|\frac{\mathrm{d}x}{x^{1/4}}\int_{-\infty}^{\infty}|\tau|\frac{e^{\lambda\tau}}{\sqrt{|\sinh\pi\tau|}}\,\mathrm{d}\tau<\infty,\quad\lambda\in\left(0,\frac{\pi}{2}\right).$$

Hence by Fubini's theorem (15) becomes

$$F_1(t)=\frac{t}{\pi i\left(1+t^2\right)^{1/2}}\lim_{\lambda\to\frac{\pi}{2}-}\int_{-\infty}^{\infty}\tau e^{\lambda\tau}\,\left(t+\left(1+t^2\right)^{1/2}\right)^{i\tau}\int_0^1 K_{i\tau}(x)\left[\frac{?(x)}{x}-\frac{2}{1+x}\right]\mathrm{d}x\,\mathrm{d}\tau. \tag{17}$$

We split the latter integral with respect to τ on two integrals over $(-\infty,M]$, $[M,\infty)$, where $M>0$ is a large fixed number. Hence we treat the integral over $(-\infty,M]$, appealing again to inequality (16) in order to justify a passage to the limit under the integral sign when $\lambda\to\frac{\pi}{2}-$ by virtue of the absolute and uniform convergence with respect to $\lambda\in[0,\pi/2]$. Therefore passing to the limit under the integral sign, we obtain that it vanishes when $t\to+\infty$ owing to the Riemann–Lebesgue lemma since the integrand belongs to $L_1(-\infty,M]$. Further, the integral over $[M,\infty)$ can be written in the form

$$I(t) = \frac{t}{\pi i(1+t^2)^{1/2}} \lim_{\lambda \to \frac{\pi}{2}-} \int_M^\infty \tau e^{\lambda\tau} \left(t + \left(1+t^2\right)^{1/2}\right)^{i\tau} \left[KL_0(\tau) - KL_1(\tau)\right] \mathrm{d}\tau, \tag{18}$$

where we denoted by

$$KL_j(\tau) = \int_0^\infty K_{i\tau}(j+x) \left[\frac{?(j+x)}{j+x} - \frac{2}{1+j+x}\right] \mathrm{d}x, \quad j = 0, 1 \tag{19}$$

modified shifted Kontorovich–Lebedev transforms of the Minkowski question mark function. But the function $f(x) = ?(x) - \frac{2x}{1+x}$ is continuous on $[0, \infty)$, $f(x) = O(x)$, $x \to 0+$ and via functional equation (4) and elementary series expansions we get

$$\frac{?(x)}{x} - \frac{2}{1+x} = 2\left(\frac{1}{x} - \frac{1}{1+x}\right) - \frac{?(1/x)}{x} \sim 2\left(\frac{1}{x^2} - \frac{1}{x^3}\right) + O\left(\frac{1}{x^4}\right), \quad x \to +\infty.$$

Consequently, employing asymptotic formula (9) with $\beta = \frac{\pi}{2}$ we derive

$$KL_0(\tau) = (2\pi)^{3/2} e^{-\pi\tau} \left[P^{-3/2}_{i\tau-1/2}(0) - P^{-5/2}_{i\tau-1/2}(0)\right] + O\left(e^{-\pi\tau/2}\tau^{-3}\right), \quad \tau \to +\infty. \tag{20}$$

On the other hand, a straightforward application of the same technique developed in [6,9,8] will drive us to an asymptotic formula of the shifted Kontorovich–Lebedev transform $KL_1(\tau)$. Indeed, since

$$\frac{?(1+x)}{1+x} - \frac{2}{2+x} = \frac{2}{(1+x)(2+x)} - \frac{?\left(1/(1+x)\right)}{1+x} \sim 2\left(\frac{1}{x^2} - \frac{3}{x^3}\right) + O\left(\frac{1}{x^4}\right), \quad x \to +\infty,$$

then $KL_1(\tau) = (2\pi)^{3/2} e^{-\pi\tau} [P^{-3/2}_{i\tau-1/2}(0) - 4P^{-5/2}_{i\tau-1/2}(0)] + O(e^{-\pi\tau/2}\tau^{-3})$, $\tau \to +\infty$. Consequently, we find

$$KL_0(\tau) - KL_1(\tau) = -\frac{6\pi\, e^{-\pi\tau/2}}{\tau(i\tau-1)(i\tau-2)} + O\left(e^{-\pi\tau/2}\tau^{-3}\right), \quad \tau \to +\infty. \tag{21}$$

Substituting this value into the latter integral in (18), we immediately conclude its absolute and uniform convergence with respect to $\lambda \in [0, \pi/2]$. Thus

$$I(t) = \frac{6\, it}{\left(1+t^2\right)^{1/2}} \int_M^\infty \frac{\left(t + \left(1+t^2\right)^{1/2}\right)^{i\tau}}{(i\tau-1)(i\tau-2)} \mathrm{d}\tau + o(1) = O\left(\frac{1}{\log t}\right) + o(1) \to 0, \quad t \to +\infty. \tag{22}$$

Therefore $F_1(t) = o(1)$, $t \to +\infty$. Combining with (11) we get the result and complete the proof of the theorem. □

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