C. R. Acad. Sci. Paris, Ser. I 349 (2011) 511–514

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex Analysis

A conformal mapping example

Un exemple d'application conforme

Wayne Smith [a], Alexander Volberg [b]

[a] *Department of Mathematics, University of Hawaii, Honolulu, HI 96822, USA*
[b] *Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA*

ARTICLE INFO

Article history:
Received 16 March 2011
Accepted 19 April 2011
Available online 5 May 2011

Presented by Gilles Pisier

ABSTRACT

An example is constructed of two Riemann maps φ and ψ of the unit disk onto the same domain such that φ'/ψ' is bounded but not bounded away from zero. This is shown by producing explicit analytic expressions of φ and ψ.

RÉSUMÉ

On construit un exemple de deux applications conformes φ et ψ du disque unité sur le même domaine telles que le rapport φ'/ψ' soit borné et le rapport ψ'/φ' non borné. On donne pour cela des expressions analytiques explicites pour φ, ψ.

Version française abrégée

Le but de cette Note est de construire un exemple d'application conforme lié à la Conjecture de Brennan [2]. La Conjecture de Brennan est une conjecture importante dans la théorie des fonctions univalentes qui concerne le degré d'intégrabilité de la dérivée d'une application conforme g d'un domaine simplement connexe planaire G sur le disque unité $\mathbb{D}$ de $\mathbb{C}$. Cette conjecture dit que

$$\int_G |g'|^t \, \mathrm{d}A < \infty \tag{1}$$

pour $4/3 < t < 4$. Il y a des classes d'applications conformes pour lesquelles la conjecture est démontrée, voir [1] par exemple. Un progrès important a été réalisé dans [4]. On trouvera dans [6] une formulation équivalente de la Conjecture de Brennan en termes d'opérateurs de composition agissant sur certains espaces de Hilbert de fonctions analytiques sur G.

Posons $\mathrm{d}\mu_p = |g'|^{2p+2} \, \mathrm{d}A$ pour un réel fixé p, et soit

$$L^2_a(\mu_p) := \left\{ F \in \mathcal{H}(G): \ \|F\|^2 = \int_G |F|^2 \, \mathrm{d}\mu_p < +\infty \right\}$$

où $\mathcal{H}(G)$ est l'espace de toutes les fonctions analytiques sur G. Alors $L^2_a(\mu_p)$ est un espace de Hilbert. La Conjecture de Brennan est équivalente à l'affirmation que pour chaque $p \in (-1/3, 1)$ il y a une application conforme τ de G sur lui-même, telle que l'opérateur de composition C_τ soit compact sur $L^2_a(\mu_p)$. L'étude des opérateurs de compositions sur $L^2_a(\mu_p)$

E-mail addresses: wayne@math.hawaii.edu (W. Smith), volberg@math.msu.edu (A. Volberg).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.04.010

conduit à une question très naturelle : *Est-il vrai que les opérateurs C_τ et $C_{\tau^{-1}}$ engendrés par l'application conforme $\tau : G \to G$ et son inverse sont bornés simultanément* ? Nous répondons ici par la négative.

1. Introduction

The purpose of this note is to construct a conformal mapping example related to Brennan's Conjecture. Brennan's Conjecture [2] is an important conjecture in univalent function theory that concerns integrability of the derivative of a conformal map g of a simply connected planar domain G onto the unit disk $\mathbb{D}$. The conjecture is that, for all such G and g,

$$\int_G |g'|^t \, \mathrm{d}A < \infty \tag{2}$$

holds for $4/3 < t < 4$. Here $\mathrm{d}A$ is area measure on the plane. There some classes of G for which it is solved, for example, in [1] Brennan's conjecture is proven for any G which is a component of Fatou's set of any second degree polynomial. In [6], V. Matache and the first author gave an equivalent formulation of Brennan's Conjecture in terms of composition operators acting on certain Hilbert spaces of analytic functions on G.

For a fixed real number p and $\mathrm{d}\mu_p = |g'|^{2p+2}\,\mathrm{d}A$, let

$$L^2_a(\mu_p) := \left\{ F \in \mathcal{H}(G){:}\ \|F\|^2 = \int_G |F|^2 \, \mathrm{d}\mu_p < +\infty \right\}$$

where $\mathcal{H}(G)$ is the space of analytic functions on G. Then $L^2_a(\mu_p)$ is a Hilbert space and it is easily checked that a different choice of the Riemann map g from G onto $\mathbb{D}$ results in the same space of functions and an equivalent norm. An analytic selfmap τ of G induces the linear *composition operator* C_τ on $\mathcal{H}(G)$, defined by $C_\tau f = f \circ \tau$.

There are interesting and promising reformulations of Brennan's conjecture, see, for example, [4] or [6]. In particular, it was shown in [6] that Brennan's Conjecture is equivalent to the statement that for each $p \in (-1/3, 1)$ and simply connected domain G, there is an analytic selfmap τ of G such that C_τ is a compact operator on $L^2_a(\mu_p)$. This result then led the authors to initiate a study of composition operators in this setting. A natural question was left open: *If the composition operator induced by an automorphism τ of G is bounded on $L^2_a(\mu_p)$, then is its formal inverse, the composition operator induced by τ^{-1}, also bounded*? This was shown to be equivalent to the following question in univalent function theory:

Let φ and ψ be Riemann maps of $\mathbb{D}$ onto the same simply connected domain G. If $\varphi'/\psi' \in H^\infty$, does it follow that $\psi'/\varphi' \in H^\infty$?

Here we construct an example that shows the answer to this question is "No". To have such a strange example the analytic expressions for such conformal map should be slightly unusual. Another approach would be a geometric one: finding the right shape of G. These approaches essentially complement each other.

First we describe a geometric point of view. We learned it from S. Rohde. Recall that a Siegel disk of a polynomial f is a simply connected component G of the Fatou set of f such that $f(G) = G$ and f is analytically conjugate to an irrational rotation of the disk. Herman [5] has shown that there exist polynomials of the form $f(z) = z^n + c$ that have a Siegel disk G with a critical point of f on the boundary of G; see also [8]. Since f on G is conjugate to a rotation, G must be bounded and so f' is bounded on G. Now let ψ be a conformal map of $\mathbb{D}$ onto G, and put $\varphi = f \circ \psi$. Then φ is also a conformal map of $\mathbb{D}$ onto G, and $\varphi'/\psi' = f' \circ \psi$ is bounded on $\mathbb{D}$. But $\psi'/\varphi' = 1/f' \circ \psi$ is unbounded since f has a critical point on the boundary. The example of Herman is not very simple, and it does not give the analytic expression for conformal maps under consideration. On the other hand, it is "visual" as one can find the pictures of Herman's Siegel disk in the literature. However, Siegel disks with critical point on the boundary are sophisticated things. Our example below is quite "tame" and direct. In our example it is quite easy to see the formulae for the required univalent maps φ, ψ. Our mechanism of "uninvertibility" is totally different, and in a sense much more simple.

The elementary example that will be presented below in Theorem 3.1 is fundamentally different. In our example the Riemann maps φ and ψ are related by a parabolic automorphism of the disk, i.e. $\psi^{-1} \circ \varphi$ is automorphism of $\mathbb{D}$ with a single fixed point that lies on the unit circle, whereas in the example coming from Herman's work $\psi^{-1} \circ \varphi$ is a rotation of the disk.

2. Preliminaries

We need to collect some facts about the regularized Hilbert transform $\mathcal{H}f$ of a function $f \in L^\infty(\mathbb{R})$, defined by the formula

$$\mathcal{H}f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} f(t) \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) \mathrm{d}t,$$

see [3, p. 110]. The term $t/(t^2+1)$ in the kernel assures convergence at infinity, but at the price that $\mathcal{H}f$ does not commute with translation. However, we do have that

$$\mathcal{H}f(x+a) - \mathcal{H}f_a(x) = C(a), \quad \text{for all } x \in \mathbb{R}, \tag{3}$$

where $f_a(t) = f(t+a)$ and

$$C(a) = \int_{-\infty}^{\infty} f(t)\left(\frac{t}{t^2+1} - \frac{t-a}{(t-a)^2+1}\right) \mathrm{d}t.$$

We now introduce the function $b : \mathbb{R} \to \mathbb{R}$ defined by

$$b(t) = (1-t) \quad \text{if } 0 \leqslant t \leqslant 1 \quad \text{and} \quad 0 \quad \text{otherwise}.$$

Proposition 2.1. *Let $u : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$u(t) = 0 \quad \text{if } t < 1 \quad \text{and} \quad b(t-n) \quad \text{for } n \leqslant t < n+1,\ n = 1, 2, 3, \ldots,$$

and let $v = \mathcal{H}u$. Then

(a) $v \in \mathrm{BMO}(\mathbb{R})$;
(b) $v(x+1) - v(x) \sim \pi^{-1} \log|x|$ *as* $x \to 0$;
(c) $v(x+1) - v(x) \leqslant B < \infty$ *for some constant B and for all $x \in \mathbb{R}$.*

It is well known that the Hilbert transform of a bounded function is in BMO, which is statement (a). Next, let u_1 be the function $u_1(t) = u(t+1)$, and notice that $u_1 - u = b$. From (3) we get that

$$v(x+1) - v(x) = \mathcal{H}u_1(x) - \mathcal{H}u(x) + C(1) = \mathcal{H}b(x) + C(1).$$

Away from 0, b is Lipschitz and so $\mathcal{H}b$ is continuous; see for example [3, Section III.1]. It is readily verified that the jump discontinuity of size 1 of b at 0 means that $\mathcal{H}b(x) \sim \pi^{-1} \log|x|$ as $x \to 0$, and that $\mathcal{H}b$ approaches a finite limit at infinity. Statements (b) and (c) follow from these observations.

Recall that the Bloch space $\mathcal{B}$ consists of analytic functions f on $\mathbb{D}$ such that

$$\bigl|f(0)\bigr| + \sup_{z \in \mathbb{D}} \bigl(1 - |z|^2\bigr)\bigl|f'(z)\bigr| = \|f\|_{\mathcal{B}} < \infty.$$

A final ingredient is the following proposition, which will be used in next section to show a function constructed for the example is univalent.

Proposition 2.2. *Let ψ be analytic and locally univalent in the unit disk $\mathbb{D}$. If $\|\log \psi'\|_{\mathcal{B}} \leqslant 1$, then ψ is univalent in $\mathbb{D}$.*

This proposition is an immediate consequence of the *Becker univalence criterion*:

Theorem 2.3. *(See [7, Theorem 1.11].) Let f be analytic and locally univalent in the unit disk $\mathbb{D}$. If*

$$\bigl(1 - |z|^2\bigr)\left|z \frac{f''(z)}{f'(z)}\right| \leqslant 1, \quad z \in \mathbb{D},$$

then f is univalent in $\mathbb{D}$.

3. Main result

Theorem 3.1. *There exist Riemann maps φ and ψ from the unit disk $\mathbb{D}$ onto the same domain G such that φ'/ψ' is bounded but ψ'/φ' is unbounded.*

Let $f = v - iu$, where u and v are the functions given in Proposition 2.1. Then $f \in \mathrm{BMO}(\mathbb{R})$ and extends to be analytic on the upper half plane Π. Let $\sigma(z) = i(1-z)/(1+z)$, so σ is a conformal map of $\mathbb{D}$ onto Π. It follows that $f \circ \sigma \in \mathrm{BMOA}(\mathbb{D})$; see [3, Corollary VI.1.3]. Since $\mathrm{BMOA}(\mathbb{D})$ is contained in the Bloch space $\mathcal{B}$, we can choose $\delta > 0$ sufficiently small so that $g = \delta(f \circ \sigma)$ satisfies $\|g\|_{\mathcal{B}} < 1$. Let ψ be the holomorphic function on $\mathbb{D}$ determined by the conditions that $\psi(0) = 0$ and $\psi' = \exp(g)$. Then Proposition 2.2 tells us that ψ is univalent.

Let τ be the automorphism of $\mathbb{D}$ defined by $\tau(z) = \sigma^{-1}(\sigma(z) + 1)$, and let $\varphi = \psi \circ \tau$. Then, with $w = \sigma(z)$, we have

$$\psi'(z) = \exp\bigl(g \circ \sigma^{-1}(w)\bigr) = \exp\bigl(\delta f(w)\bigr),$$

and similarly

$$\psi'\bigl(\tau(z)\bigr) = \exp\bigl(g \circ \sigma^{-1}(w+1)\bigr) = \exp\bigl(\delta f(w+1)\bigr).$$

Hence

$$\frac{\varphi'(z)}{\psi'(z)} = \frac{\psi'(\tau(z))\tau'(z)}{\psi'(z)} = \exp\bigl[\delta\bigl(f(w+1) - f(w)\bigr)\bigr]\tau'(z).$$

Since τ is an automorphism of $\mathbb{D}$, τ' and $1/\tau'$ are bounded. Hence, using Proposition 2.1(c), we have

$$\bigl|\varphi'(z)/\psi'(z)\bigr| = \exp\bigl[\delta\bigl(v(w+1) - v(w)\bigr)\bigr]\bigl|\tau'(z)\bigr| \leqslant \exp(\delta B)\bigl\|\tau'\bigr\|_\infty.$$

Similarly, we see from Proposition 2.1(b) that

$$\bigl|\psi'(z)/\varphi'(z)\bigr| = \exp\bigl[-\delta\bigl(v(w+1) - v(w)\bigr)\bigr]/\bigl|\tau'(z)\bigr|$$

is unbounded.

References

[1] K. Baranski, A. Volberg, A. Zdunik, Brennan's conjecture and the Mandelbrot set, Int. Math. Res. Notices (IMRN) (1998) 589–600.
[2] J.E. Brennan, The integrability of the derivative in conformal mapping, J. London Math. Soc. (2) 18 (1978) 261–272.
[3] J. Garnett, Bounded Analytic Functions, Academic Press, 1981.
[4] H. Hedenmalm, S. Shimorin, Weighted Bergman spaces and the integral means spectrum of conformal mappings, Duke Math. J. 127 (2) (2005) 341–393.
[5] M.-R. Herman, Are there critical points on the boundaries of singular domains, Comm. Math. Phys. 99 (4) (1985) 593–612.
[6] V. Matache, W. Smith, Composition operators on a class of analytic function spaces related to Brennan's conjecture, Complex Anal. Oper. Theory, doi:10.1007/s11785-010-0090-5, in press.
[7] Ch. Pommerenke, Boundary Behavior of Conformal Maps, Springer-Verlag, 1992.
[8] J.T. Rogers Jr., Critical points on the boundaries of Siegel disks, Bull. Amer. Math. Soc. 32 (3) (1995) 317–321.