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Algebraic Geometry

A Note on vector bundles on Hirzebruch surfaces

Une Note sur des fibrés vectoriels sur des surfaces de Hirzebruch

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ABSTRACT

In literature, two basic construction methods have been used to study vector bundles on a Hirzebruch surface. On the one hand, we have Serre's method and elementary modifications, describing rank-2 bundles as extensions in a canonical way (Brînzănescu and Stoia, 1984 [4,5], Brînzănescu, 1996 [6], Brosius, 1983 [7], Friedman, 1998 [9]), and on the other hand, we have a Beilinson-type spectral sequence (Buchdahl, 1987 [8]). Morally, the Beilinson spectral sequence indicates how to recover a bundle from the cohomology of its twists and from some sheaf morphisms (the differentials of the sequence). The aim of this Note is to show that the canonical extension of a rank-2 bundle can be deduced from the Beilinson spectral sequence of a suitable twist, called *the normalization*. In the final part we give a cohomological criterion for a topologically trivial vector bundle on a Hirzebruch surface to be trivial. To emphasize the relations and the differences between these two construction methods mentioned above, two different proofs are given.

RÉSUMÉ

Dans la littérature, deux méthodes de construction fondamentales ont été utilisées pour étudier les fibrés vectoriels sur une surface de Hirzebruch. D'une part, nous avons la méthode de Serre et les modifications élémentaires, décrivant d'une manière canonique les fibrés de rang deux comme des extensions (Brînzănescu et Stoia, 1984 [4,5], Brînzănescu, 1996 [6], Brosius, 1983 [7], Friedman, 1998 [9]) et d'autre part, nous avons la suite spectrale de Beilinson (Buchdahl, 1987 [8]). Moralement, la suite spectrale de Beilinson nous indique comment récupérer un fibré à partir de la cohomologie de ses tensorisations et de certains morphismes de faisceaux (les différentielles de la suite spectrale). Le but de cette Note est de montrer que l'extension canonique d'un fibré de rang deux peut être déduite de la suite spectrale de Beilinson d'une tensorisation convenable, appellée *la normalisation*. Dans la dernière partie, nous donnons un critère cohomologique pour qu'un fibré vectoriel topologiquement trivial sur une surface de Hirzebruch soit trivial. Afin de souligner les relations et les différences entre les deux méthodes de construction mentionnées ci-dessus, deux démonstrations différentes sont présentées.

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We fix the notation. Let $X = \Sigma_e \xrightarrow{\pi} \mathbb{P}^1$ be a Hirzebruch surface, $e \geqslant 0$. Denote by C_0 the negative section, i.e. $C_0^2 = -e$, and by F a fiber of the ruling.

1. Extensions and Beilinson spectral sequences

In this section, we compare two basic construction methods used to study vector bundles on a Hirzebruch surface: extensions and Beilinson spectral sequences.

1.1. Rank-2 bundles as extensions [4–7,9]

Besides the Chern classes, any rank-two vector bundle M on the Hirzebruch surface X has two numerical invariants describing it as an extension in a canonical manner, [4–6]. The first invariant d_M is defined by the splitting type on a general fiber F: if $M|_F \cong \mathcal{O}_F(d) \oplus \mathcal{O}_F(d')$ with $d \geqslant d'$, then $d_M := d$. The second invariant r_M is obtained from a push-forward. Note that $\pi_*(M(-dC_0))$ is either of rank one or two, according to whether $d > d'$ or $d = d'$. If $d > d'$, we put $r_M := r = \deg(\pi_*(M(-dC_0)))$. If $d = d'$, then $\pi_*(M(-dC_0)) = \mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(s)$ with $r \geqslant s$ and we put $r_M := r$. The integer s provides us with an extra-invariant for bundles with $d = d'$ (of "equal" type, following the terminology of [7]).

The bundle M is expressed as an extension

$$0 \to \mathcal{O}_X(dC_0 + rF) \to M \to \mathcal{O}_X\big(d'C_0 + r'F\big) \otimes \mathcal{I}_\zeta \to 0, \tag{1}$$

where $\zeta \subset X$ is a zero-dimensional subscheme, and this extension is unique if either $d > d'$ or $d = d'$ and $s < r$. The twist $M(-dC_0 - rF)$ is called the *normalization* of M, and M is called *normalized* if $d = r = 0$. Hence the extension (1) is unique if and only if the space of global sections of the normalization is one-dimensional.

The length of ζ is the *uniformity* degree of M. Precisely, M has the same splitting type over all the fibers of π (i.e. it is uniform) if and only if ζ is empty, [1]. The invariant r measures the stability of M, see [2].

The discriminant of M is computed by the formula:

$$\frac{1}{4}\Delta(M) = c_2(M) - \frac{c_1(M)^2}{4} = \ell(\zeta) - \frac{1}{4}\big(d - d'\big)\big(e\big(d' - d\big) - 2\big(r' - r\big)\big), \tag{2}$$

therefore, if $d = d'$, $\Delta(M) = 4\ell(\zeta) \geqslant 0$. Alternatively, in the case $d = d'$, M can be described as an elementary modification of a projectively flat rank-2 bundle [7,9]:

$$0 \to \pi^*\big(\pi_* M(-dC_0)\big)(dC_0) \to M \to Q \to 0 \tag{3}$$

where $Q(-dC_0)$ is supported on (possibly multiple) fibers passing through ζ, is of total degree $-\ell(\zeta)$, and has no global sections. Note that $\pi^*(\pi_* M(-dC_0))(dC_0)$ is split, isomorphic to $\mathcal{O}_X(dC_0 + rF) \oplus \mathcal{O}_X(dC_0 + sF)$, and in particular, any vector bundle M of splitting type $\mathcal{O}_F(d)^{\oplus 2}$ over each fiber of π is decomposable.

Proposition 1. *Notation as above.*

1. *If either $d > d'$ or $d = d'$ and $s < r$, then the bundle $\mathcal{O}_X(dC_0 + rF)$ coincides with the image of the multiplication map*: $H^0(X, M(-dC_0 - rF)) \otimes \mathcal{O}_X(dC_0 + rF) \to M$.
2. *If $d = d'$ and $s = r$, then the bundle $\pi^*(\pi_* M(-dC_0))(dC_0)$ coincides with the image of the multiplication map*: $H^0(X, M(-dC_0 - rF)) \otimes \mathcal{O}_X(dC_0 + rF) \to M$.

Proof. Applying the projection formula, we may assume, after a twist, that M is normalized, i.e. $d = r = 0$. In this case, it suffices to identify the image of the evaluation morphism $H^0(\pi_*(M)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \pi_* M$. If $d' = c_1(M) \cdot F < 0$ then $\pi_*(M) = \mathcal{O}_{\mathbb{P}^1}$. If $c_1(M) \cdot F = 0$ then $\pi_*(M) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(s)$, and hence the image of the evaluation morphism is either $\mathcal{O}_{\mathbb{P}^1}$ or $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ according to whether $s < 0$ or $s = 0$. $\square$

The reduction to the normalization, used in the proof of the previous proposition, will appear again in the sequel in relation with Beilinson spectral sequences.

1.2. Beilinson spectral sequences, following Buchdahl ([8], see also [3])

The diagonal Δ of any Hirzebruch surface $X = \Sigma_e$ inside $X \times X$ can be described scheme-theoretically as the zero-locus of a global section in a rank-two vector bundle over $X \times X$ [8]. In other words, X satisfies the *diagonal property* [11]. This description represents the foundation of the Beilinson spectral sequence and is achieved in two steps. Put $Y = X \times_{\mathbb{P}^1} X \subset X \times X$, and denote by $p_1, p_2 : X \times X \to X$ the two projections. Firstly, consider the embedding $\Delta \subset Y$ and observe that $\mathcal{O}_Y(\Delta) = (p_1^* T_{X|\mathbb{P}^1}(-C_0) \otimes p_2^* \mathcal{O}_X(C_0))|_Y$ [8,3]; recall that $T_{X|\mathbb{P}^1}(-C_0) = \mathcal{O}_X(C_0 + eF)$ and $p_1^* \mathcal{O}_X(F)|_Y \cong p_2^* \mathcal{O}_X(F)|_Y$. Secondly, use an extension lemma to pass from the fibered product to the usual product. Precisely, there exists a rank-two bundle G on $X \times X$, given by a non-trivial extension

$$0 \to p_1^* T_{X|\mathbb{P}^1}(-C_0) \otimes p_2^* \mathcal{O}_X(C_0) \to G \to \mathcal{O}_{X\times X}(Y) \to 0 \tag{4}$$

and a global section of G whose zero-scheme coincides with $\Delta \subset X \times X$ (see, for example, [3]). In particular, we obtain a truncated Koszul complex

$$0 \to \wedge^2 G^* \to G^* \to \mathcal{O}_{X\times X} \to 0, \tag{5}$$

which is exact except at $\mathcal{O}_{X\times X}$. Let M be a vector bundle of arbitrary rank on X. Twisting the complex (5) by $p_2^*(M)$ and taking the hypercohomology, we obtain a spectral sequence abutting to M [8]:

$$E_1^{p,q} = R^q p_{1*}\big(\wedge^{-p} G^* \otimes p_2^*(M)\big) \quad \Rightarrow \quad \begin{cases} M & \text{if } p+q=0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $E_1^{p,q} = 0$ if $p \notin \{-2,-1,0\}$ or if $q \notin \{0,1,2\}$. The remaining terms of the spectral sequence are computed by twisting the dual of the extension (4) by $p_2^*(M)$ and applying p_{1*} [8,3]. For any q we have:

$$E_1^{0,q} \cong H^q(X,M) \otimes \mathcal{O}_X, \qquad E_1^{-2,q} \cong H^q\big(X, M(-C_0-F)\big) \otimes \mathcal{O}_X\big(-C_0-(e+1)F\big),$$

and $E_1^{-1,q}$ can be determined from the exact sequence

$$H^q\big(X, M(-F)\big) \otimes \mathcal{O}_X(-F) \to E_1^{-1,q} \to H^q\big(X, M(-C_0)\big) \otimes \mathcal{O}_X(-C_0-eF). \tag{6}$$

Morally, using the Beilinson spectral sequence, any vector bundle should be completely determined by the cohomology of suitable twists and by some vector bundle morphisms (the differentials of the spectral sequence). To illustrate this principle we show that in the rank-2 case, the extensions of the previous section can be recovered from the Beilinson spectral sequences of the normalizations. For simplicity, we assume that the bundle in question is already normalized, in which case it will either have $h^0 = 1$ or $h^0 = 2$, according to the two cases of Proposition 1.

Theorem 1. *Let M be a normalized rank-2 vector bundle on a Hirzebruch surface X.*

1. *If $h^0(X,M) = 1$, then either $E_\infty^{-1,1} = 0$ or $E_\infty^{-2,2} = 0$ and the filtration provided by the Beilinson spectral sequence coincides with the canonical extension* (1)*.*
2. *If $h^0(X,M) = 2$, then $E_\infty^{-2,2} = 0$ and the filtration provided by the Beilinson spectral sequence coincides with the defining elementary modification sequence* (3)*.*

Proof. Note that the natural map $E_1^{0,0} \cong H^0(X,M) \otimes \mathcal{O}_X \twoheadrightarrow E_\infty^{0,0} \subset M$ coincides with the evaluation map.

Suppose $h^0(X,M) = 1$. From the hypothesis and Proposition 1, it follows that $E_1^{0,0} \cong E_\infty^{0,0} \cong \mathcal{O}_X$ is the first term of the sequence (1).

By the shape of the spectral sequence, $E_\infty^{-2,2} \subset E_1^{-2,2} \cong H^2(X, M(-C_0-F)) \otimes \mathcal{O}_X(-C_0-(e+1)F)$, hence $E_\infty^{-2,2}$ is either torsion-free of rank one or it is zero.

If $E_\infty^{-2,2} = 0$, then the Beilinson filtration reduces to

$$0 \to E_\infty^{0,0} \to M \to E_\infty^{-1,1} \to 0,$$

and, since $E_\infty^{0,0} = \mathcal{O}_X$ and $h^0(X,M) = 1$, it follows that this exact sequence coincides with the extension (1). Note that this situation occurs if $c_1(M) \cdot F = 0$, as $E_1^{-2,2} = 0$ in this case.

If $E_\infty^{-2,2}$ is torsion-free of rank one, then it is a quotient of M by a rank-one subsheaf, hence $E_\infty^{-1,1}$ must be zero and the Beilinson filtration reduces to the sequence

$$0 \to E_\infty^{0,0} \to M \to E_\infty^{-2,2} \to 0,$$

which coincides again, from the hypothesis and Proposition 1, with the extension (1). This situation occurs, for example, if $M = \mathcal{O}_X \oplus \mathcal{O}_X(-C_0-F)$ on $X = \mathbb{P}^1 \times \mathbb{P}^1$.

The case $h^0(X,M) = 2$ is solved in a similar manner. □

As already mentioned, if M is a normalized rank-2 bundle with $h^0(X,M) = 1$, then the canonical extension (1) is unique, and hence it only depends on M. If $h^0(X,M) = 2$, i.e. M is an elementary modification along fibers of the trivial bundle, uniqueness of (1) fails, however, the defining elementary modification sequence is unique in this case.

2. The cohomological description of trivial bundles

In this section we give a cohomological criterion for a topologically trivial vector bundle on a Hirzebruch surface to be trivial. Recall that, for Hirzebruch surfaces, topological triviality is equivalent to the vanishing of the Chern classes.

Theorem 2. *Notation as above. A topologically trivial vector bundle M on X of rank $\geqslant 2$ is trivial if and only if $h^0(X, M(-C_0)) = h^0(X, M(-F)) = h^1(X, M) = h^2(X, M(-C_0 - F)) = 0$.*

Proof. (Arbitrary rank, using the Beilinson spectral sequence.) It is clear that the trivial bundle satisfies the four vanishing conditions.

Conversely, we suppose that the bundle M has $h^0(X, M(-C_0)) = h^0(X, M(-F)) = h^1(X, M) = h^2(X, M(-C_0 - F)) = 0$ and we prove it is trivial. To this end, we use the Beilinson spectral sequence.

By topological triviality, the Chern classes of M vanish, and by Riemann–Roch, we obtain $\chi(M) = r$ and $\chi(M(-C_0)) = \chi(M(-F)) = \chi(M(-C_0 - F)) = 0$. It implies, using the hypothesis, the sequence (6) and the Serre duality that $E_1^{-2,q} = 0$ and $E_1^{-1,q} = 0$ for all q. Beside, $E_1^{0,0} \cong \mathcal{O}_X^r$ and $E_1^{0,1} = E_1^{0,2} = 0$.

Hence $E_\infty^{p,q} = 0$ for all $(p,q) \neq (0,0)$, and it follows that $M \cong E_\infty^{0,0} \cong \mathcal{O}_X^r$. $\square$

Proof. (Rank-2, using the canonical extension.) We prove that M is normalized. Using formula (2), it will imply that M is trivial.

First, we verify that $d_M = 0$. Suppose $d_M = d > 0$. From the extension (1) and from the condition $h^0(M(-C_0)) = 0$ it follows that $r_M = r < 0$. Since $h^1(M) = 0$, the long cohomology sequence implies that $h^1(\mathcal{O}_X(dC_0 + rF)) = 0$, i.e. $r \geqslant de - 1$ which is possible only if $e = 0$ and $r = -1$. On the other hand, the formula (2) implies $d(2r - de) = -2d \geqslant 0$, contradiction. Hence $d_M = 0$ and the extension (1) is of type

$$0 \to \mathcal{O}_X(rF) \to M \to \mathcal{O}_X(-rF) \to 0,$$

where $r = r_M$. From the definition of r_M it follows that $r \geqslant 0$. Since $h^0(M(-F)) = 0$, r must be $\leqslant 0$. Hence $r_M = 0$. We have used three out of the four vanishing conditions from the hypothesis. Note that, since $M \cong M^*$ in this case, the condition $h^2(X, M(-C_0 - F)) = 0$ follows from the other conditions, and the Serre duality. $\square$

Remark 1. Note that in the statement of Theorem 2 in the rank-2 case, no assumption was made on the splitting type of M on fibers, however, a posteriori, the bundle is normalized. It follows also from the proof that a topologically trivial rank-2 bundle is analytically trivial if and only if it is normalized.

Remark 2. The classical Beilinson spectral sequence [10] for $\mathbb{P}^2$ provides us with a similar triviality criterion. Precisely, a flat vector bundle of rank $\geqslant 2$ on $\mathbb{P}^2$ is trivial if and only if $h^0(M(-1)) = h^1(M(-2)) = 0$. Other results in the same spirit can be proved for bundles on rational scrolls (use [3]), and higher-dimensional projective spaces.

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